Symmetry: Geometry and Elementary Group Theory

Cooper Union Summer Research Internship

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1. Introduction

Over the course of six weeks, the mathematics group for summer research internship has explored various ways to connect two branches of mathematics – geometry and algebra. Along with this main goal in mind, we constructed many projects to further our understanding of both. In Project I, we represented the algebra portion in terms of complex numbers. For Project II, we represented algebra by using permutations. For both projects, however, group theory was involved. Group theory is abstract algebra, and can be composed of various types of operations. In order to be deemed a group, four properties had to be satisfied. These properties were closure, associativity, identity, and inverse. These terms were essential to our studies. Inside groups contained subgroups, these helped us pair algebra with their geometric counterparts. For the geometry portion, we investigated regular polygons and polyhedra (2-D and 3-D shapes that have sides with equal length). We furthered our understanding through concepts such as tiling planes and duality (a regular polyhedra inscribed in another regular polyhedra). Our projects led us to study the patterns of geometric symmetries in great depth.

Symmetry is a rotation or a reflection of a geometric object that sends it back to its original phase. The movement is performed around a specific axis through a vertex, edge, or face. Polygons and polyhedra have multiple symmetries. Symmetries helped us view algebra more concretely, therefore leading us into the study of geometry. For Project I, we explored the symmetries within 2-D figures, regular polygons. Their symmetries consisted of both rotations and reflections. For Project II, we studied the symmetries of regular polyhedra, which consisted solely of rotations.

2. Groups

A group is a set of any mathematical symbols that includes four basic properties under a certain operation. These properties are closure, associativity, identity and inverses. All of these properties must be applicable to the set under the given operation in order for it to be considered a group. In this research project we used the subgroups to combine sets of symmetries of shapes as well as polyhedra.

Group G under operation *

i. Closure: if x and y are in G then x*y is also in G
ii. Associativity: if x, y, and z are in G then (x*y)*z = x*(y*z)
iii. Identity: there is an element e in the group such that x*e = x
iv. Inverses: x*y = e when y= x^{-1}

The first property that we looked at was closure. Closure states that if you have two elements in a group then the composition of the two elements under the given operation is also in the group. For example if you had a group under addition and the elements A and B then A+B is also in the group. This has proven to be difficult to show for some operations.

The next property that a group must consist of is associativity. In plain words associativity shows that it does not matter in what order you do an operation. An example of this can be seen when you have a group under addition that consists of the elements A, B, and C then $(A+B)+C = A+(B+C)$.

The third property is that of the identity. The identity is element that if composed with another element in the group under the given operation then you will get that other element. For example in the case of addition zero is the identity because $A+0 = A$. in the case of multiplication one is the identity as $A \times 0 = A$

The final property is inverses. Inverses state that for every element in G there is another element that when composed under * gives you the identity. For example in a group under addition $A+B = 0$ so that $B = A^{-1} = -A$

The simplest example of a group is all integers. We denoted the group of all integers as Z. ($Z = \{\dots -3, -2, -1, 0, 1, 2, 3\dots\}$)

This is a group because it follows all the properties of a group (where a, b, and c are in Z)

i. Closure: an integer plus an integer is always an integer. If a and b are in Z then so is $a+b$
ii. Associativity: $(a+b)+(c) = a+(b+c)$
iii. Identity: 0 is the identity because $a+0 = a$
iv. Inverses: $a+(-a) =0$

Although integers are a group under addition they are not a group under multiplication because the group does not satisfy all the properties. It fails the property of inverses.

Integers under multiplication

i. Closure: if a and b are in Z then so is axb

ii. Associativity: ax(bxc) = (axb)xc

iii. Identity: 1 is the identity because ax1=a

iv. Inverses: ax 1/a = 1

As you can see inverses fails because the inverse of an integer is a fraction, which is not an integer. Also under multiplication zero does not have an inverse

For this project we will be using groups with elements of the rotations of regular shapes and regular polyhedra.

Subgroups:

A subgroup is a subset of a group that also contains the four properties (closure, associativity, identity, and inverses). A subset is a list of elements that are present in a group but don't necessarily contain the four properties. In order to show that a subset is a subgroup you must first show that the elements are present in the group and then you must show that that subset has all four properties.

An example of this is that the even integers are a subgroup of all integers. Because they have all four properties under addition

Even integers are a subgroup of integers under addition. This is true because even integers are a subset of integers as even numbers can be denoted as sums of integers.

An even integer is denoted as $2Z = \{2n \mid n \text{ is in } Z\}$

i. Closure: if 2a, an even integer and 2b, an even integer are in 2Z then so is 2(a+b) which is also an even integer
ii. Associativity: $2a+(2b+2c)=(2a+2b)+2c$
iii. Identity: $2a+0 =2a$
iv. Inverses: $2a+2(-a)=0$

Although even numbers are a subgroup of integers under addition, odd numbers are not. Odd numbers are a subset of integers as they can be denoted in terms of integers.

An odd integer is denoted as $2Z+1 = \{2n+1 \mid n \text{ is in } Z\}$

i. Closure: if 2a+1, an odd integer and 2b+1, an odd integer are in 2Z+1 then their sum 2(a+b+1) should be in 2Z+1, but it is not as this is an even integer.
ii. Associativity: $[(2a+1)+(2b+1)]+(2c+1) = (2a+1)+[(2b+1)+(2c+1)]$
iii. Identity: there is no identity because in the case of addition zero should be the identity but zero is considered an even integer.
iv. Inverses: the inverse of 2a+1 is 2(-a) -1

Order:

Order has two meanings in the sense of group theory. The first definition states the order of a group, which is in plainest terms how many elements are in the group. Order is written by using the absolute value symbol. (if H is a group then the order of H is written as $|H|$). For example: in this project we looked at the rotational symmetries of a triangle, which preserve the shape. These rotations were by 0, 2π/3, and 4π/3. This is actually a group as it has all four properties. (We denoted the rotation by 0 as e, the identity)

Rotational symmetries of a triangle were the group of G={rot(e), rot(2π/3), rot(4π/3)}

So the order of G is 3 as there are only three elements. This can be written as $|G| = 3$.

One can also find the order of an element. The order of an element is how many times you have to compose that element with itself in order to get the identity. The order of an element is denoted with | as well. (if you wanted to find the order of rot(2π/3) it would be written as |rot (2π/3)|)

For example: if we take the group G as previously stated: G={rot(e), rot(2π/3), rot(4π/3)}

|rot(e)| = 1

This is because when you compose a rotation by 0 once you get the identity.

|rot(2π/3)| = 3

This is because when you compose a rotation by $2\pi/3$ three times with itself you get the identity.

$|rot(4\pi/3)| = 3$

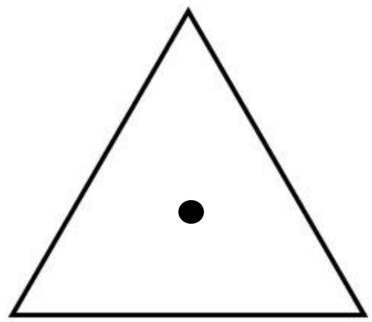
This is because, like $rot(2\pi/3)$, when you compose a rotation by $4\pi/3$ three times with itself you get the identity.

3.1 Introduction and Objectives

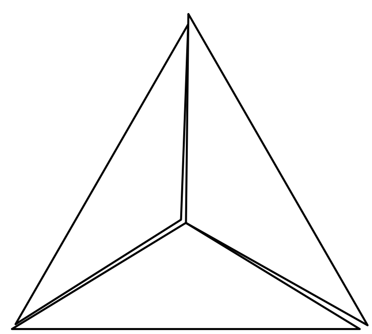
The inherent symmetry in regular shapes and polygons allows for us to study those properties in various ways. One of the most intriguing methods we can see these properties is in the context of groups. Observing these symmetries through these groups gives way to understanding the symmetries and polygons in an entirely different context – algebraically. Erecting this bridge, it becomes ever so clear the similarities in geometry and algebra. First, we must establish how these regular shapes come together in simply a geometric context, and then we can draw these shapes in terms of planes and algebra – namely complex numbers in the complex plane. The objective at hand becomes the idea that what we label these shapes becomes irrelevant, as their structures are identical and soon we can use this to reach deeper meaning in a robust and logical theory.

3.2 The Unit Circle and Regular Polygons – Geometric Symmetries: Rotations and Reflections.

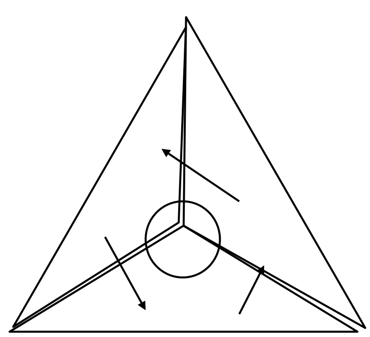
To consider the different geometric symmetries of regular polygons, let's observe concrete and simple examples, such as the equilateral triangle and the square. For these shapes, we want transformations the send the shape to itself. That is to say, when we move the points in the shape, they form the same exact shape. The first of such transformations will be a rotation about the center of the shape.



In order to send the shape to itself, we must rotate by a specific angle. In order to find this angle, we have to look at smaller triangles within our larger triangle.



Each of these are smaller isosceles triangles that form a circle at their vertex. We can use this to find out how much we should rotate by.



As a result this also splits up the angles inside the circle by three. Since there are 360° in a circle, or 2π, we divide this by 3 to give us how much we need to rotate by to send these interior triangles to the next, and thus the whole triangle to itself: $2\pi/3$

This pattern also holds for other regular polygons for the same reason. A rotation that sends a square to itself would be $2\pi/4$ or just $\pi/2$. Thus in general for any polygon with *m* number of sides (or an m-gon) the angle we would rotate by would be: $2\pi/m$.

But this is not the only rotation; we can rotate by this angle again to reach a different rotation. We shall say that we can perform this *k* times. Thus rotating a triangle twice would result in a rotation of $2*2\pi/3$ or just $4\pi/3$. Rotating a square twice would result in $4\pi/4$ or just π. In general this results in rotations by $2\pi k/m$.

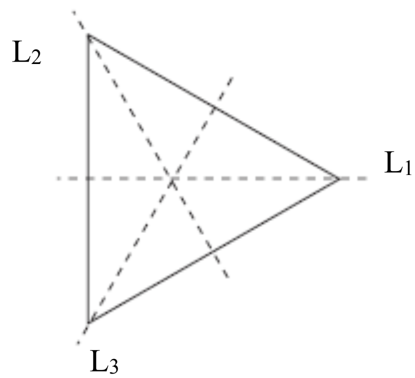
We can rotate by this infinitely many times, but after a certain number of rotations; we reach our original triangle, where the vertices are in the same place, essentially doing no rotation at all. This is known as the identity, because it is the exact same triangle we had to start with, with the vertices being in exactly the same place. This happens when our rotations bring us to full circle or 2π. For a triangle this would be $6\pi/3$, and for a square, $8\pi/4$. We notice that we can split this us as $3*2\pi/3$ and $4*2\pi/4$. This is when k and m are equal to each other. Now we know what we limit our rotations to.

Let P_m denote a regular m-gon.

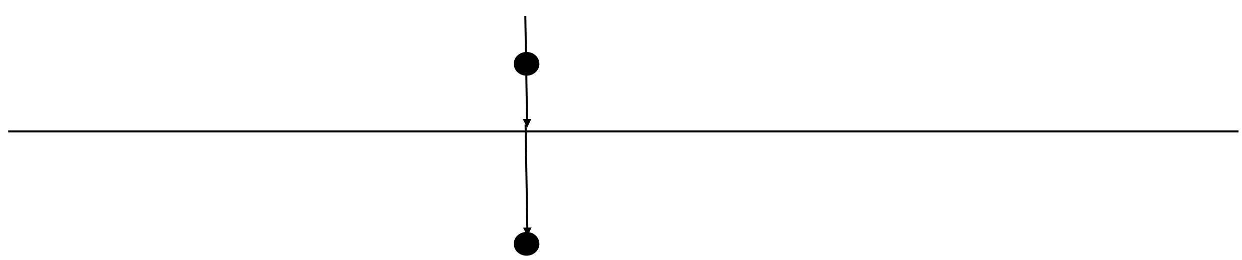
$$\mathrm{Rot}(P_m) = \{\mathrm{Rot}(2\pi k/m), 0 \leq k < m\}$$

Note here that when k = 0, it is the same as not rotating at all, thus being the identity.

Now we have to define our second type of symmetry: a reflection. Reflections in our polygons will always be across a line. Let's draw these lines for our triangle and label them.



Line 1 (denoted as L_1) will be the line that passes through the first vertex, line 2 (denoted L_2) will be the line that passes through the second vertex, and line 3 (denoted L_3) will be the line that passes through the third vertex. A reflection through line 1 will send a point to that line to another point an equal distance from the line. For a single point it looks like this:



For a triangle, a reflection across that line will do it for every point in that triangle. For example, a reflection across line 1 will send the second vertex to the third vertex and the third vertex to the second vertex. A reflection across line 2 will send the first vertex to the third vertex and vice versa. And point on the line gets fixed there, since the point it gets moved to it itself. For a triangle, this means that there will be three lines of symmetry. For a square, there will be 4 and so on.

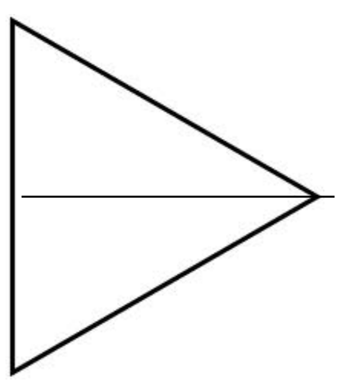
Now it is important to note that all the rotations can be seen as one rotation multiple times. For triangles, that is a rotation by $2\pi/3$, for squares it will be $2\pi/4$, and for regular pentagons it will be $2\pi/5$.

Let r denote a rotation by $2\pi/3$.

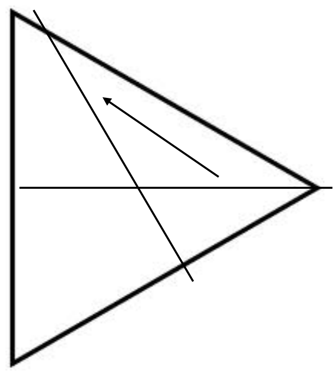
$r \circ r = r^{1+1} = r^2$

r^2 in this case will be a rotation by $2(2\pi/3)$ or just $4\pi/3$, which we have seen is another rotation for a triangle. This works for any m-gon and you will get all the rotations of that m-gon. To get the identity it is simply r^0 which will be a rotation by $0(2\pi/m) = 0$.

We can say something similar for the reflections. Let's observe one reflection in our triangle.

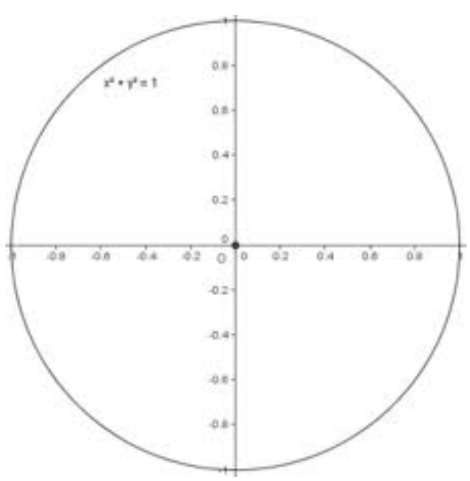


This line represents our baseline reflection, meaning all the reflections will come from some form of this one reflection, just like the rotations. Now in order to achieve the other reflections, one simply rotates that one reflection by r, when r denotes a rotation by $2\pi/3$. This gives us our second line of reflection.

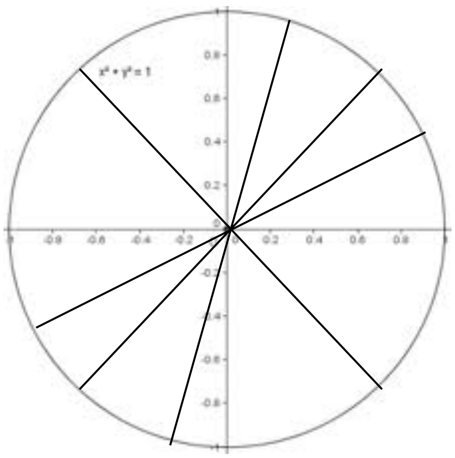


Our first line goes go our second line by that rotation, and we can do it once more to get all the lines of symmetry in a triangle. We can see that all the lines of symmetry can be achieved by a composition of one reflection and all the rotations of that m-gon. Thus, we have now developed all the symmetries of regular m-gons in terms of one rotation and a composition of one reflection and those rotations.

Our next shape to define is the unit circle. This circle exists on a plane, for example the real plane. Our unit circle is a circle that has a radius of 1 and its center is on the origin. It looks like this:



This circle also shares the same types of symmetries as our regular m-gons. A rotation about the center of the circle will send the circle to itself. To imagine this, just a spin a circle on its center and it will be the same circle no matter how much you spin it by. Similarly, any line of reflection across the origin will send the circle to itself. For example these are all possible lines of reflections in the unit circle:



3.3 Complex Numbers and the Complex Plane

The solution to the quadratic formula $x^2 + 1=0$ ends with $x= \sqrt{(-1)}$. With a little thought, it is realized that there exists no real solution. When two numbers of the same sign are multiplied, the result will always be a positive. The square root of -1 is referred to as an imaginary number, and has been labeled to the constant i. This opens up the group of numbers known as the complex number, of which real numbers are a subgroup. Any complex number is written under the form a+bi, where a and b are real numbers. This divides the complex number into two parts: the real part, a, and the imaginary part bi. Thus, any real number x is also a complex number, under the form x+0i.

The complex plane is a way to graph complex numbers, and provides a useful tool for rotations and reflections. It works similarly to the real plane, but the x-value is replaced with a and the y-value is replaced with b. so, 6+5i becomes what on the real plane would be (6,5).

Multiplication of complex numbers is very much the same as multiplying binomials. For instance,

$$(a+bi)(x+yi)$$

$$=(ax+by(i^2)+ayi+xbi)$$

$$=(ax-by+(ay+xb)i)$$

For example:

$$(2+5i)(3-7i)$$

$$=(6+(-35)(\ i^2)+(-14)\ i+15i)$$

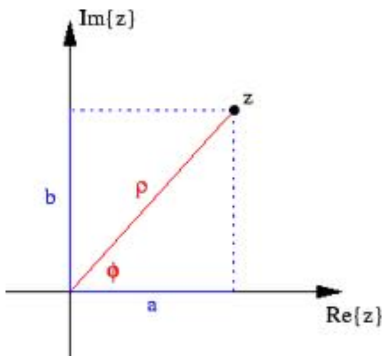
$$=(6-(-35)+(-14+15)\ i)$$

$$=41+(1)i$$

$$=41+i$$

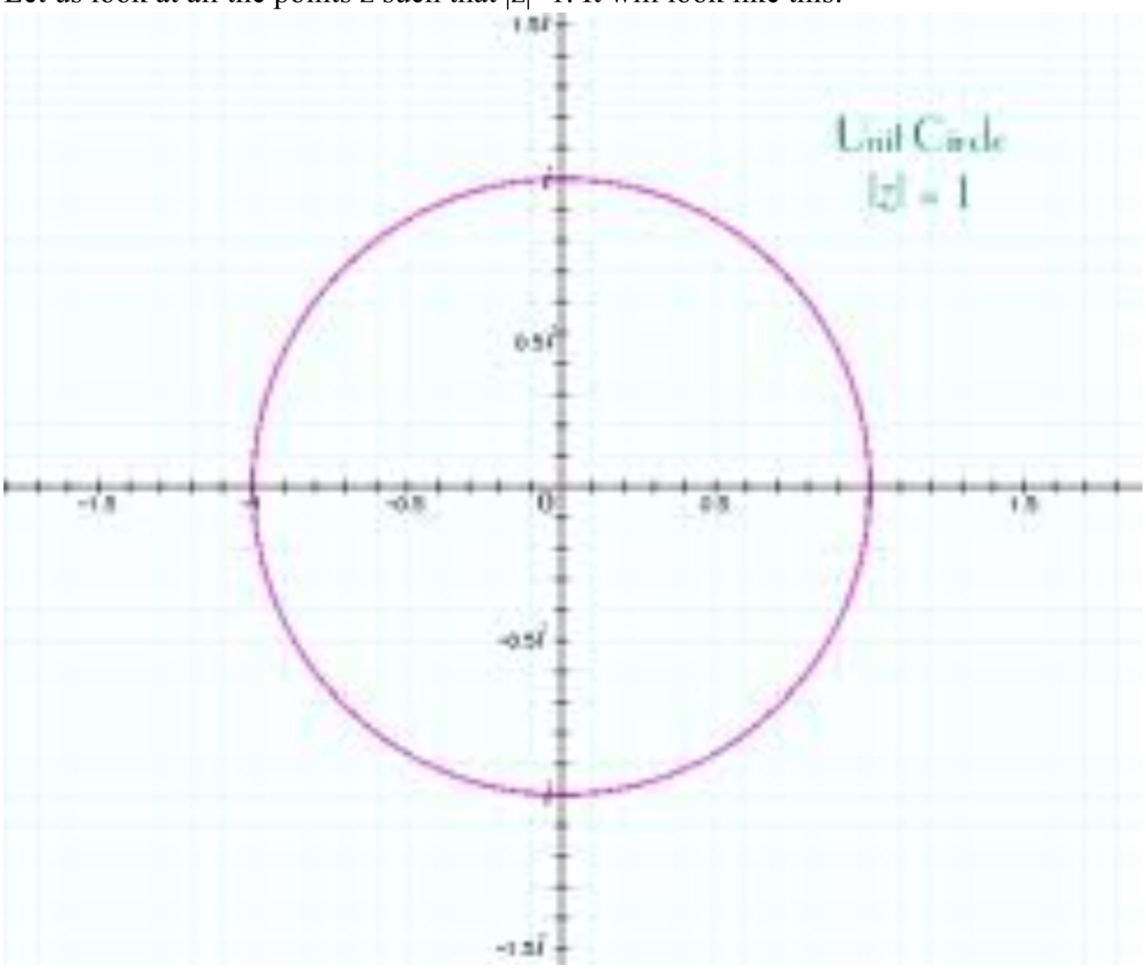
The Conjugate of a complex number z=a+bi is denoted , where = a-bi. Note that, were one to compare this value to the previous value in the real plane, he would find (x,y) and (x,-y). in other words, the process by which z transforms into is through a reflection over the Y axis, or Ref (0).

The Norm of a complex number z is denoted |z|, where $|z| = \sqrt{(a^2+b^2)}$. Looking closer, if |z| =ρ, then:



In other words, |z| is the Euclidian distance between z and the origin.

Let us look at all the points z such that |z|=1. It will look like this:



In other words, this collection of z forms the unit circle. Let us define this:

= {z ϵ **C** | |z| = 1}

3.4 Rotational Symmetries and Complex Multiplication

One of the best things about the complex plane is it allows us to view rotations algebraically. Since every point in the unit circle is written a+bi, and in the real plane, any point on the unit circle is labeled (sinΘ, cosΘ), any point in the complex unit circle is written z=cosΘ +isinΘ.

Now, let us take a look at the results of rotation.

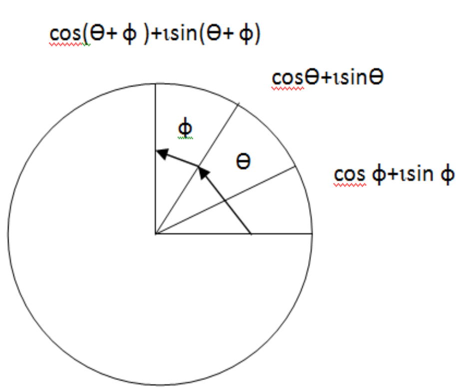
if you take z_1 = cosΘ+isinΘ, and rotate it by ϕ, ideally the newest point would be z_2= cos(Θ+ ϕ) +isin(Θ+ ϕ). But, if we express the rotation of ϕ as its own point (so a Rot(ϕ) becomes z_3=cos ϕ +isin ϕ) and multiply them together, we get:

$$(\cos\Theta+i\sin\Theta)(\cos \phi+i\sin \phi)$$

$$=(\cos\Theta \cos \phi+(i^2) \sin\Theta \sin \phi +i\sin \phi \cos\Theta+i\sin \phi \cos\Theta)$$

$$=(\cos\Theta \cos \phi- \sin\Theta \sin \phi+i2\sin \phi \cos\Theta)$$

$$= \cos(\Theta+ \phi)+i\sin(\Theta+ \phi)$$



In other words, the function of rotation becomes multiplication between two points, allowing us to express these functions algebraically, and not just geometrically.

3.4.1 The Unit Circle as a Group

Let $w = \cos \Theta + i\sin\Theta$, and let M_w be the map which multiplies each z in S^1 by w. using this, we can prove that this notation is the same as Rot (Θ):

$$M_w=wz$$

$$\text{If } w=(\cos\Theta+i\sin\Theta) \text{ and } z=(\cos \phi+i\sin \phi),$$

$$\text{Then } wz= \cos(\Theta+ \phi)+i\sin(\Theta+ \phi)$$

$$\text{Therefore, } M_w = \mathrm{Rot}(\Theta).$$

Using this, we can, for example, find out what multiplying by i does to a complex number:

$$M_i$$

$$i=0+1i$$

$$i=\cos()+ i\sin()$$

$$M_i=\mathrm{Rot}()$$

Now, let's notice something about its group structure, namely that $M_{wx}=M_{wx}$

$$w,x,z \in S^1$$

$$M_{wx}z=(wx)z$$

$$M_{wx}=w(xz)$$

By the associativity of the group S^1, $M_{wx}=M_{wx}$

Due to the proofs above, we can say that S^1, through M_w, can be paired with the Rotations of S^1. And, because $M_{wx}=M_{wx,}$ we can say that their group structure is the same. Therefore, S^1 and Rot (S^1) are Isomorphic-we have just shown a connection between algebra and geometry.

3.5.1 Subgroups and Inscribed Polygons

Now that we know that rotation is multiplication, it should be obvious that Rot(Θ) ◦ z, where z=cos Θ+ isinΘ, will go to the spot z^2, or cos2Θ + isin2Θ. Now, let us define a new subgroup, U_m.

Let $U_m= \{z \in \mathbf{C} \mid z^m = 1\}$.

Now, let us prove two things:

Prove Z is in U_m iff Θ = , $\{k \in \mathbf{Z}\}$

If z is in $U_{m,}$

$$z^m= \cos(2) + i\sin(2)$$

$$z^{(m\ x\ )}=\cos() + \imath\sin()$$

$$z= \cos() + \imath\sin()$$

From this we can learn that any z inside of U_m has the form cos() + ısin().

Let us define ζ_m as cos(2/m) + ısin(2/m).

Now, let’s relate ζ_m to U_m:

Let $<_m>$ be the cyclic subgroup generated by $_m$

$$<_m> = \{{}_m^k \mid 1 \leq k < m, k \in \mathbf{Z}\}$$

Since anything in U_m is in the form ,

$$<_m> = U_m$$

Now we can prove that the various values of z in U_m form the vertices of a regular polygon:

If 1, ζ_m, and ζ_m^n,

$$1 \text{ x } \zeta_m = \zeta_m$$

$$\zeta_m \text{ x } \zeta_m^n = \zeta_m^{n+1}$$

$$1 \text{ x } \zeta_m^n = \zeta_m^n$$

This means that $\zeta_m^n \text{x } \zeta_m = \zeta_m^{n+1}$

In other words, the difference between the two points remains constant, no matter what they are multiplied by, proving that the shape formed had sides that are the same, and therefore is regular. From now on, let us define the shape created by U_m as P_m.

Using previous notation, we can say that $M_{\zeta_m}{}^k$ = Rot().

Now, let's prove something about the group structure:

$$1 \leq k,x < m$$

$$M_{\zeta_m{}^k\zeta_m{}^x}(z)= (\zeta_m{}^k\zeta_m{}^x)z$$

$$Rot() \circ Rot()(z) = \zeta_m{}^k(\zeta_m{}^x z)$$

By the associative property of U_m,

$$M_{\zeta_m{}^k\zeta_m{}^x}(z) = Rot() \circ Rot()(z)$$

Since $M_{\zeta_m}{}^k$ is multiplication of a vertex inside (P_m), we can now conclude that Rot(P_m) and U_m are Isomorphic. We have now shown another connection between algebra and geometry, through the vertices of a polygon on the complex plane.

3.5.2 Finite Subgroups of the Circle.

Now that we know all about U_m, let's learn about its subgroups.

If U_m and U_k, and k|m,then

$$kx=m$$

$$U_m = <\zeta_m{}^y>$$

$$\zeta_m{}^y = \cos(2y/m) + \imath\sin(2/m)$$

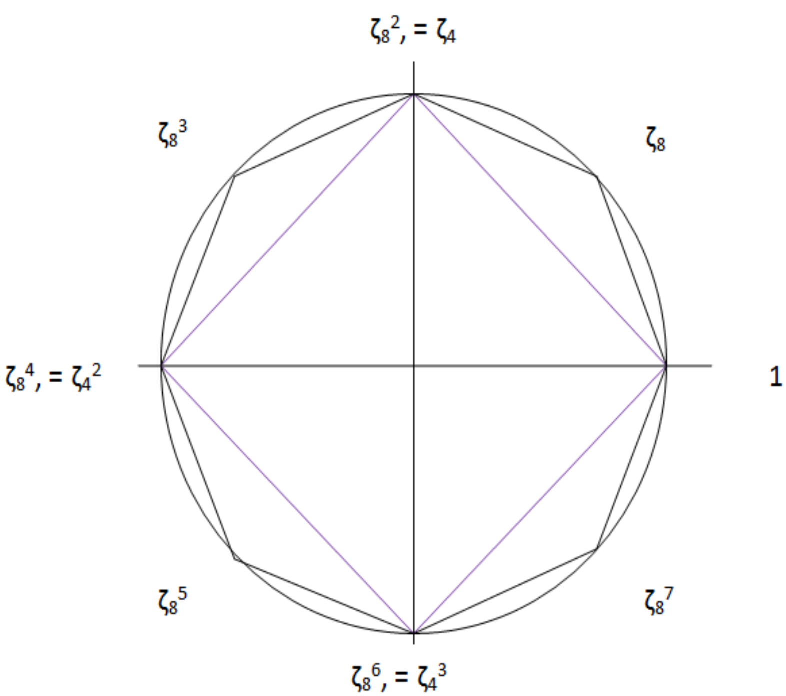
$$\zeta_k{}^w = \cos(2w/k) + \imath\sin(2/k)$$

$$\zeta_m{}^y = \zeta_{kx}{}^y$$

$$\zeta_{kx}{}^y = \cos(2y/kx) + \imath\sin(2/kx)$$

U_k will therefore be mapped to some points in U_m, which will have more points than U_k. in other words, U_k is another regular polygon inscribed inside U_m. for example:

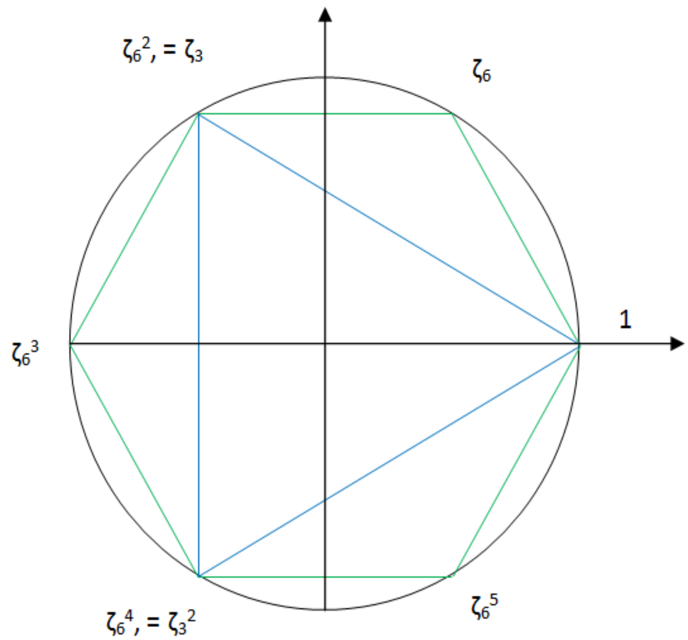
If m=8 and k=4,



Inside of the octagon formed by U_8, there is a square, the same square for U_4.

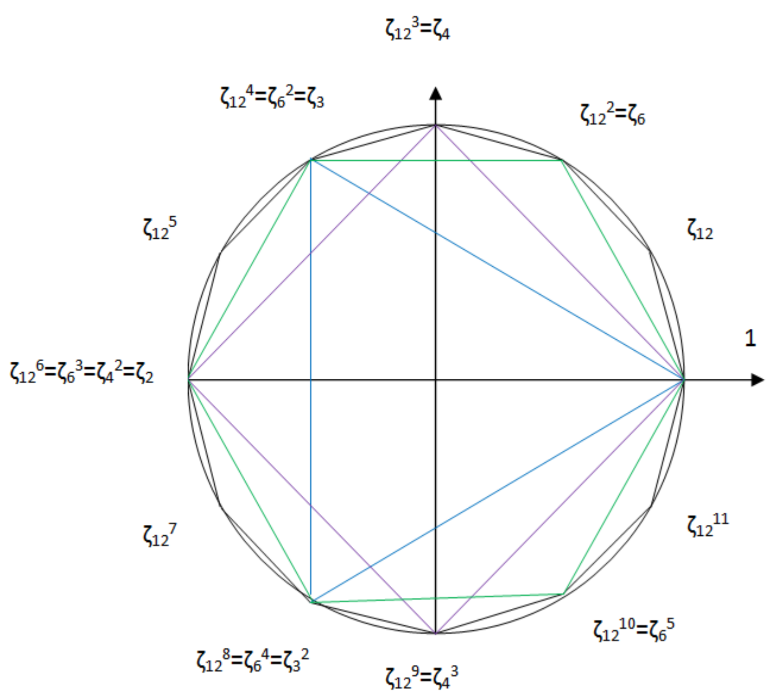
Notice that the square has exactly the same vertices as some of the octagons'. Now, let's try a different example.

If m=6 and k=3,



As you can see, a triangle, U_3, is inscribed into a larger hexagon, U_6.

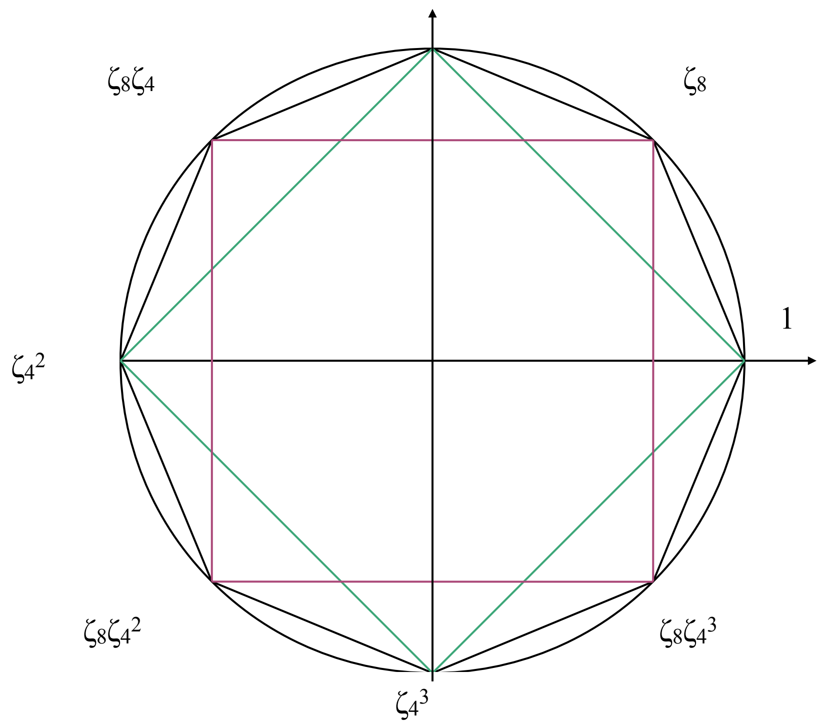
If m=12, k,m,n|m, and k=3, m=4, n=6,



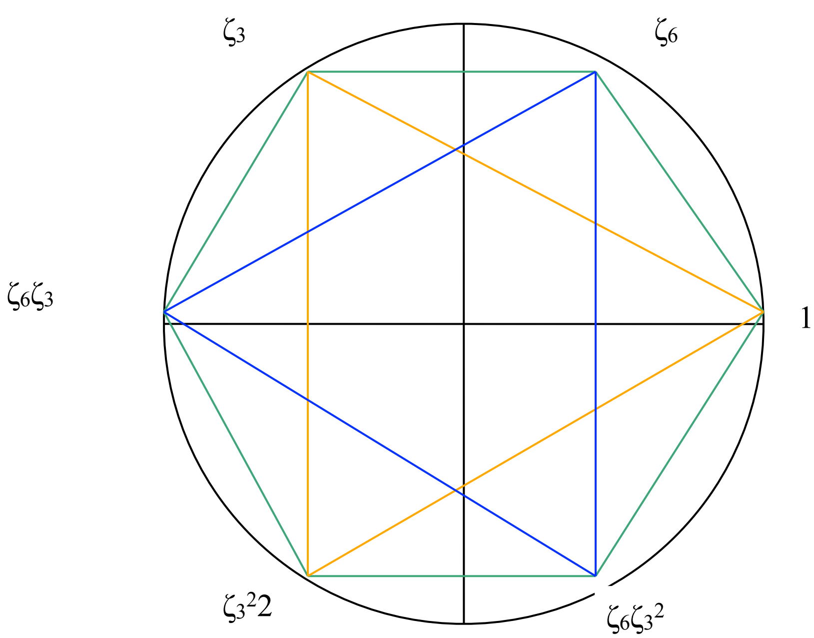
Notice something about U_8: while U_4 divides it, there are 4 remaining vertices, the connection of which would make another square. Since a rotation by ζ_8 would accomplish this, we can therefore write the set of points of this new square as $\zeta_8 U_4$, which, expanded, looks like: $\{ \zeta_8 \text{ x } 1, \zeta_8 \text{ x } \zeta_4, \zeta_8 \text{ x } \zeta_4^2, \zeta_8 \text{ x } \zeta_4^3\}$.

Let's see this in action:

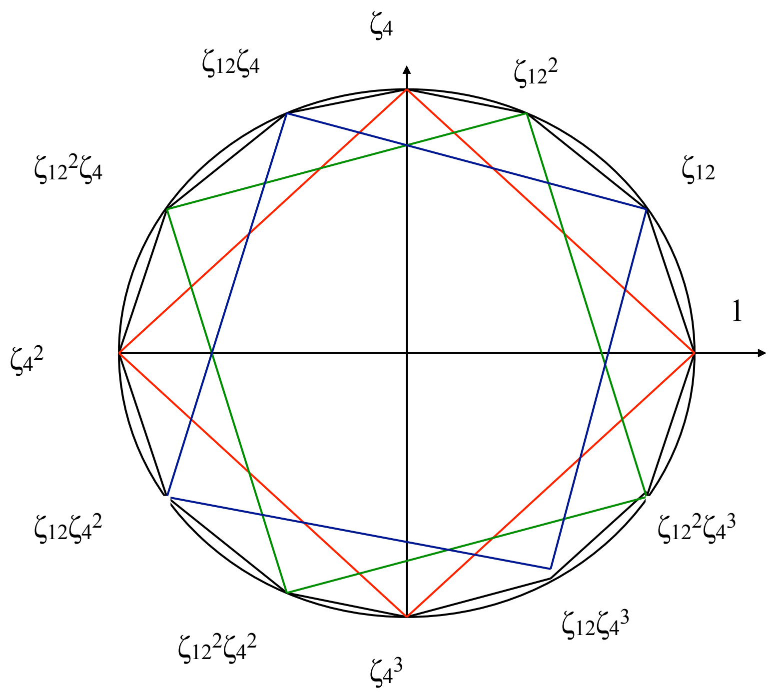
ζ_4



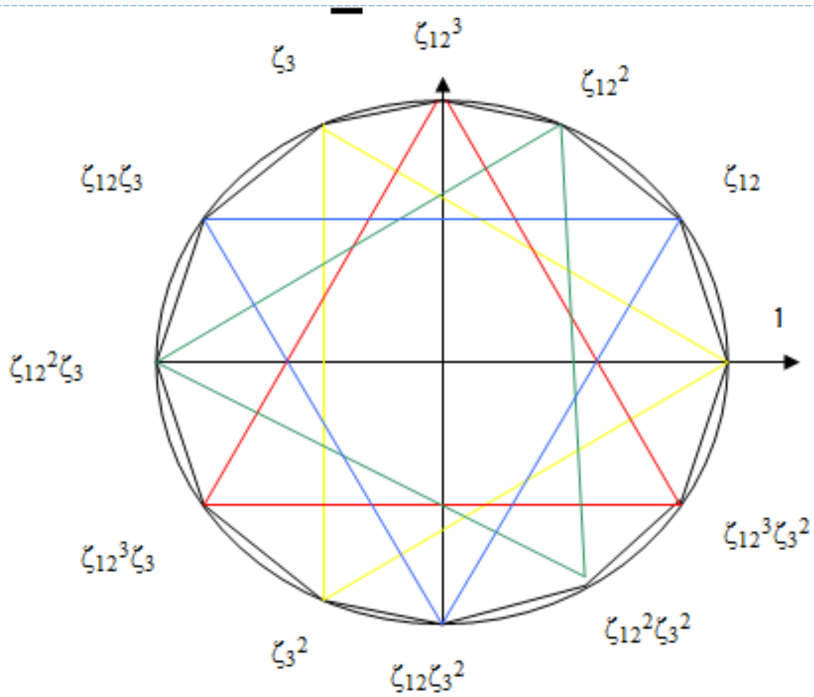
Now, let's see when m = 6 and k = 3



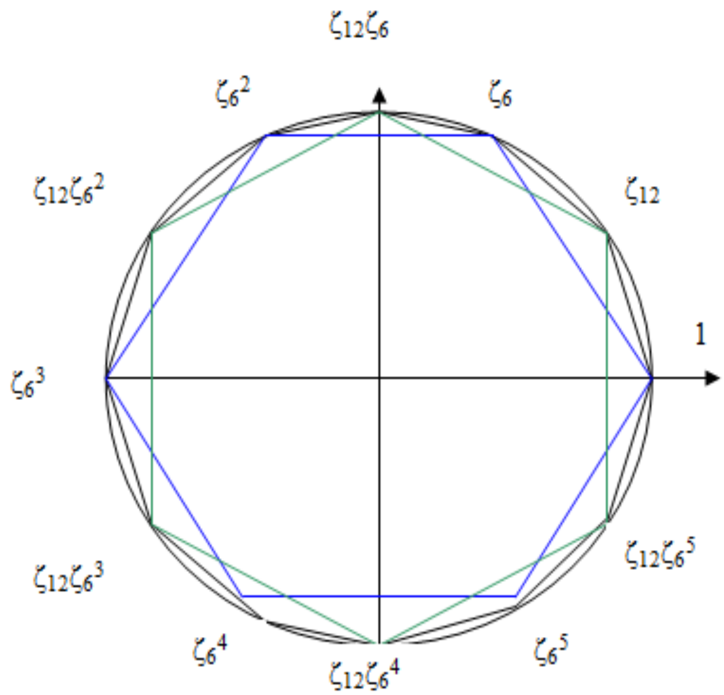
When m = 12, k = 3?



When m = 12 and k = 4?



When m=2 and k = 6?



Notice how a rotation of ζ_8 will send a k-gon to itself. Finally, let's prove that any subgroup of S^1 is, in fact, a form of U_m:

Let H be a subgroup of S^1

If h ε H, then the order of h must be finite, as H itself is finite.

Let z ε H be of maximum order m

Since z is in S^1, and belongs in a subgroup, it can be written as ζ_m^y , with y and m being relatively prime.

Therefore, $< z > = < \zeta_{m >} {}^{=}$ U_m since $z = \zeta_m^y$ and is therefore a generator of U_m . since z generates U_m , and z is in H, U_m must be inside H

Assume H is not equal to H, and suppose w is in H but not in U_m .

w must equal ζ_n^x , as it is a point in H, but cannot equal ζ_m^y , since that would put it into U_m .

since w is of the highest order, and is of the form ζ_n^x , ζ_n must be in H, as it is also of the highest order.

Since H is closed under multiplication, ζ_m x ζ_n must exist, since ζ_m actually is in H. however, this new number would have a larger order than w, making a contradiction.

From this proof, we see that any subgroup of S^1 forms a polygon. The completeness of this proof is remarkable, due to the fact that no matter what a subgroup is, it will always create a polygon, and the points will always be under the form ζ_m^y.

Through the complex plane, we have seen correlation between algebra and geometry, through rotations and rotational symmetries. In the next section, we shall see how reflections and symmetry by reflections can also be expressed algebraically.

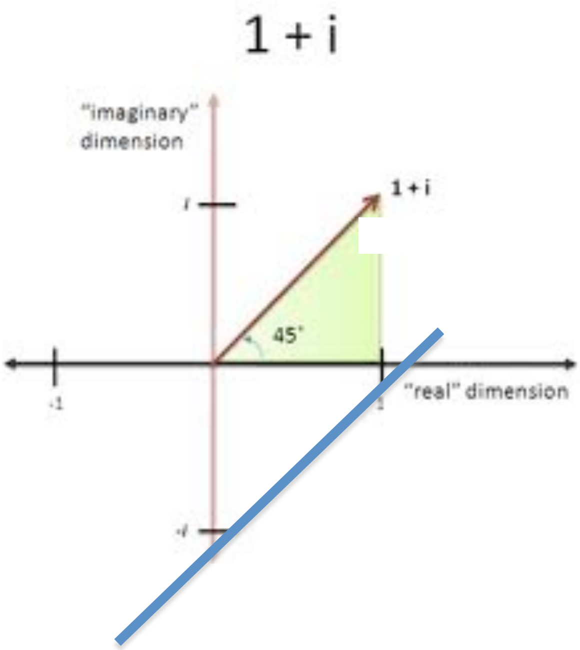
3.6 The Full Symmetry Group of the Circle: Reflective Symmetries and Complex Conjugation

As we have seen, all the finite subgroups of rotations in the unit circle are given by rotations that are multiples of $2\pi/m$ – which are the rotations of regular m-gons. Just as we explored reflections in m-gons after establishing their rotations, we can do this for the unit circle.

To do this, we must define our reflections in the complex plane. For any angle θ where 0 ≤ θ < π, we shall let L_θ denote the line passing through the origin which creates an angle θ with

the p … s means that it will create a line with an

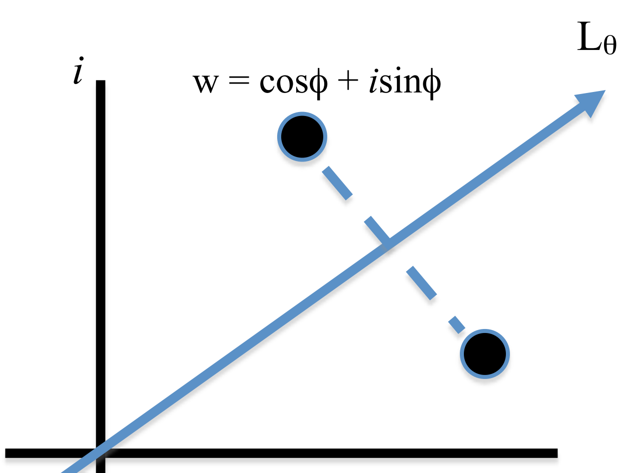
angl … x plane it looks like this:



This works exactly the same as any other reflection. A reflection through $L_{\pi/4}$ will send a point to that line to another point an equal distance from $L_{\pi/4}$. To make the notation clear, a Ref(θ) is a reflection across L_θ . Let's take make this more general.

Suppose w = cosϕ + *i*sinϕ

We want to show that Ref(θ)(w) = cos(2θ - ϕ) + *i*sin(2θ - ϕ). This would allow us to see how any point moves with any reflection.



The angle of L_θ is θ and the angle of point w is ϕ. In order to get the new point, we have to subtract ϕ from θ twice and add that to ϕ. Written out it looks like this:

$\cos(\phi + 2(\theta - \phi)) - i\sin(\phi + 2(\theta - \phi))$

Which reduces to

$\cos(2\theta - \phi) + i\sin(2\theta - \phi)$

This gives us our reflected point.

There is one reflection that is worth exploring. This is the reflection by 0, or basically the x-axis. In the real plane, a reflection by the x-axis sends a point (x,y) to the point (x,-y). In the complex plane this still applies.

So a point $z = \cos(\theta) + i\sin(\theta)$ gets sent to $\cos(\theta) - i\sin(\theta)$. But we have seen this before as the complex conjugate. So now we can state:

$\mathrm{Ref}(0)(z) = \bar{z}$

From now on, we can denote Ref(0) by

Since any reflection sends the unit circle to itself, we can now state all the symmetries of the unit circle:

$$Sym(S^1) = \{Rot(\theta), \mathrm{Re}f(\phi) \mid 0 \le \theta < 2\pi, 0 \le \phi < \pi\}$$

Now that we have laid this out, we can start composing these different symmetries. First let's compute a composition of two reflections:

$$
\begin{aligned}
&\operatorname{Re} f(\theta_1) \text{ o } \operatorname{Re} f(\theta_2) \\
&= \cos(2\theta_1 - 2\theta_2) + i\sin(2\theta_1 - 2\theta_2) \\
&= Rot(2\theta_1 - 2\theta_2) \\
&= Rot(2(\theta_1 - \theta_2))
\end{aligned}
$$

This shows us that a composition of two distinct reflections leads to a rotation.

Let's compute a composition of a rotation and reflection:

$$
\begin{aligned}
&\operatorname{Re} f(\theta_1) \text{ o } Rot(\theta_2) \\
&= \operatorname{Re} f(\theta_1) \text{ o } \cos(\theta_2) + i\sin(\theta_2) \\
&= \cos(2\theta_1 - \theta_2) + i\sin(2\theta_1 - \theta_2) \\
&= \operatorname{Re} f(2\theta_1 - \theta_2) \\
&= \operatorname{Re} f(\theta_1 - \theta_2 / 2)
\end{aligned}
$$

We can see that a composition of a rotation followed by a reflection results in a reflection, but is it the same as a reflection followed by a rotation?

$$
\begin{aligned}
&Rot(\theta_1) \text{ o } \operatorname{Re} f(\theta_2) \\
&= Rot(\theta_1) \text{ o } \cos(2\theta_2) + i\sin(2\theta_2) \\
&= \cos(\theta_1 + 2\theta_2) + i\sin(\theta_1 + 2\theta_2) \\
&= \operatorname{Re} f(\theta_1 + 2\theta_2) \\
&= \operatorname{Re} f(\theta_1 / 2 + \theta_2)
\end{aligned}
$$

The answer evidently is no, but it is still a reflection, so any combination of a rotation and reflection will lead to reflection. We can still find a way to interchange the order of the way we compose the rotations and reflections. Let's go back to , or a reflection by 0. When this is our reflection, we can find a way to interchange the order. Recall that a rotation can also be expressed as M_w, as that it what we will use in this situation.

$M_w \circ \bar{c}$
$= M_w \circ \cos(\theta) - i\sin(\theta)$
$= M_w \circ \cos(-\theta) + i\sin(-\theta)$
$= (\cos(\phi) + i\sin(\phi))(\cos(-\theta) + i\sin(-\theta))$
$= \cos(\phi - \theta) + i\sin(\phi - \theta)$
$= \cos(-\phi + \theta) - i\sin(-\phi + \theta)$
$= (\cos(\theta) - i\sin(\theta))(\cos(-\phi) + i\sin(\phi))$
$= \bar{c} \circ M_{w^{-1}}$

Since we have now defined our compositions, we can prove that Sym(S^1) is actually a group. As usual, we have to go through the 4 conditions required to be a group.

Closure: We have proved that a composition of any element of the symmetries in S^1 leads to another symmetry in S^1. More specifically, a rotation composed with a rotation leads to another rotation, a rotation and a reflection in any order leads to a reflection, and a composition of two distinct reflections leads to a rotation.

Associativity: It can be assumed from being integers.

Identity: This will be a rotation by 0:

$Rot(\theta) \circ Rot(0)$
$= Rot(\theta + 0)$
$= Rot(\theta)$

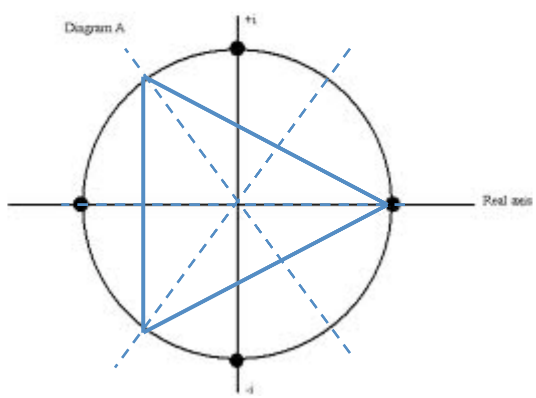
Inverse:

We can also now know that the $Rot(S^1)$ is a subgroup of $Sym(S^1)$. We know that all the rotations of S^1 are contained in all the symmetries of S^1, and we have proved that the rotations are a group, thus they are a subgroup of the symmetries of S^1.

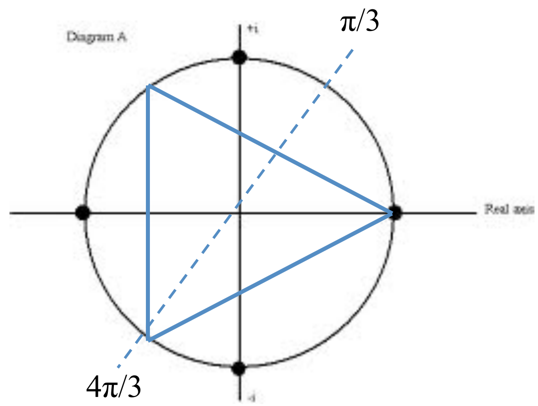
Just as we looked at all the finite rotational symmetries of S^1, we can now explore all the finite rotational and reflective symmetries of S^1, which will again be through m-gons. We shall call these dihedral groups defined as all the symmetries of S^1 that send P_m to itself.

$$Sym(P_m) = D_m$$

For example, D_3 is all the rotational and reflective symmetries of an equilateral triangle. D_4 is all the symmetries of a square, etc. Just as we did with regular m-gons, we have to determine the lines of symmetries for D_m. Let's look at D_3



These are just the lines we used for our 3-gon. Thus the angles that would form these lines are 0, $\pi/3$, and $2\pi/3$. Notice here that $4\pi/3$ is replaced by $\pi/3$. This is because in our definition of reflections, we limited our angles to be between 0 and π. But $\pi/3$ is just the other end of $4\pi/3$:



Now we can define all the reflections of P_m in S^1:

$\text{Ref}(D_m) = \{\text{Ref}(\pi k/m) \mid 0 \le k < m\}$

Just as we could express all the reflections of an m-gon as a composition of one reflection and rotations, we can do this for D_m, except we can do it in complex notation. For example:

Now we have a new way of expressing D_m

We can proceed to show that D_m is a subgroup of S^1. We know that D_m is contained in S^1, so now we have to show the 4 properties of a group.

Closure:

Association: Automatic from associativity of complex numbers.

Identity:

Inverse:

Now, we can define the subgroups of D_m. Let's have H' be all the finite rotations of S^1 and H be all finite subgroups of the symmetries of S^1. We know that H' is a subgroup of H since the all the finite rotations are contained within all the symmetries of S^1, and are a group.

The trivial case would be when H equals the identity, meaning H' must also be the identity and is of course a group in itself.

Let's consider the case where H has no reflections, this would mean that H would be all the rotations and thus be equal to H'. This means H is all the finite rotations of S^1.

Let's consider the case where H' consist only of the identity. The only rotation here is the rotations by 0, which is the identity. This forces H to be just one reflection and the identity. This is the only way that reflections can be in a group by themselves. The next case makes this a little more clear.

Lastly, let's consider the case where H' is not equal to the identity, and is of maximum order. This means that it must contain all the rotations of that order. Now there must be all the symmetries of S^1. This is because of the property of closure. H must contain reflections, and even if it was one reflection, we know that the composition of one reflection and all the rotations lead to all the reflections of that m-gon. It also cannot contain any more reflections because that would violate that closure. The compositions of the reflections lead to all the rotations, again compelled to follow closure in order to be a group.

These four cases lay out all the possible finite subgroups of S^1, which are all the symmetries of an m-gon. Explicitly:

$H = \{e\}$

$H = \{Rot(S^1)\}$

$H = \{e, Ref(\pi k/m)\}$

$H = \{D_m\}$

This theory provides us with all the finite subgroups of the unit circle. This is possible because we described the symmetries of m-gons through algebra in the complex plane. The isomorphism between the geometry of the shapes and the algebra of the unit circle in the complex plane bridged this gap to give way to our final theory and the result of all our work.

4.1. Introduction and Objectives

In Project I, we observed the geometric symmetries - the rotations and reflections, for regular polygons. We saw how distinct and precise every movement was. To return a figure back to its original image, we had to rotate with certain fractions, which were found according to the shape itself. Similarly in Project II, we will now begin to study the three dimensional rotational symmetries for regular polyhedra (the 3-D analog of regular polygons). Our objective in this section is to examine regular polyhedra in great detail, connect the 3-dimensional space with 2-dimensional planes, and to ultimately find clear ties between two seemingly different branches of mathematics, geometry and algebra. We began with discovering the only polyhedra that can be formed with regular polygons, where the same number of faces must meet at any given vertex. In addition to satisfying that rule, the figure must also be enclosed. These regular polyhedra are known as Platonic solids.

Although there are infinitely many regular polygons, there are certain measures to finding all the possible Platonic solids. We found that the only five possible Platonic solids that can be formed were the tetrahedron, cube, octahedron, dodecahedron, and icosahedron. We denoted the possible number of faces and the number of faces which meet at a vertex by using the Schläfli symbol $\{p,q\}$. A great deal is also determined by a shape's number of vertices, edges, and faces. As we constructed our Platonic solids, we also discovered that each solid contained a corresponding dual, or multiple duals. These were other Platonic solids that shared the same rotational symmetries and they were able to be inscribed in one another. Afterwards, we saw the relationship that existed between regular polygons and regular tilings of the plane.

4.2. Geometry: Platonic Solids – the Five Regular Polyhedra

A great deal of our summer research internship for mathematics revolved around the use of five particular regular polyhedra. These figures existed only in 3-D space, distinctly different from regular polygons which exist in a 2-D plane. However, both share extremely similar connections, as we will soon discover. These five regular polyhedra are deemed as Platonic solids, after the Ancient Greek philosopher, Plato. Although these figures were actually known centuries before Plato's time, the Ancient Greeks evaluated the solid figures in greatest depth. Some discrepancy has been made about the true founder of the solids. Some credit mathematicians such as Pythagoras or Theaetetus. The majority however, credit to Plato as the father of the platonic solids, given the name of the polyhedrons.

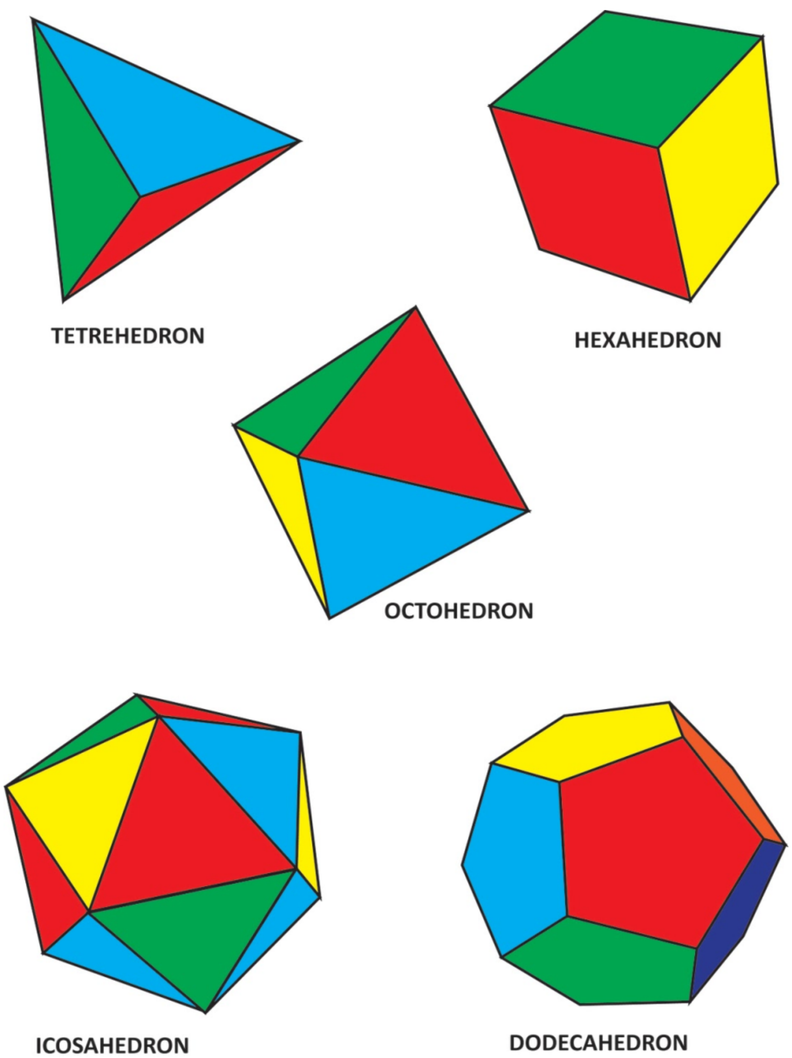


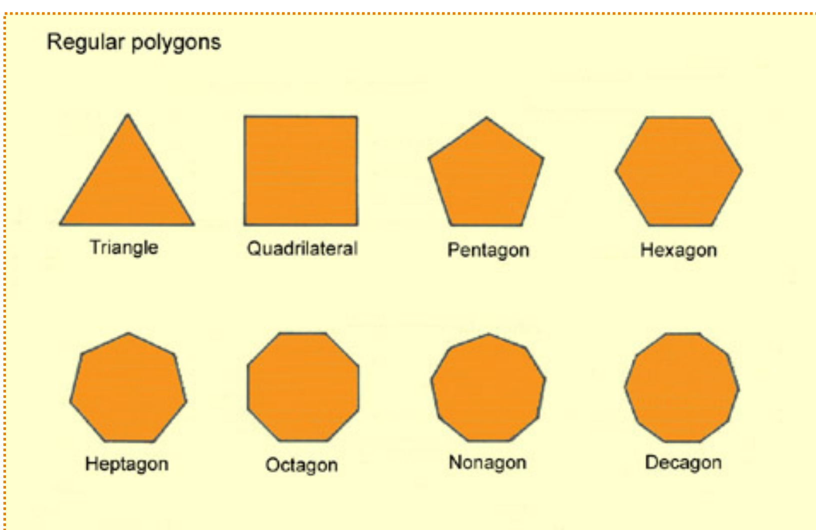
FIG-1 THE PLATONIC SOLIDS

A Platonic solid is a polyhedra that is bound with congruent regular polygons (particularly, equilateral triangles, squares, and equilateral pentagons). In order to be deemed a Platonic solid, the same number of faces must meet at every given vertex, and the figure must be enclosed.

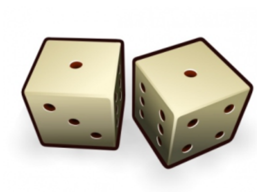
The five Platonic solids consist of a tetrahedron, cube (also known as hexahedron), octahedron, dodecahedron, and icosahedron. All these figures consist of a pattern, distinct duals, and a concrete number of rotational symmetries for each figure (which we are able to express both algebraically and geometrically).

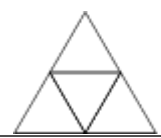
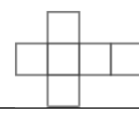
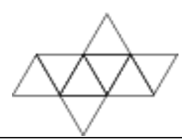
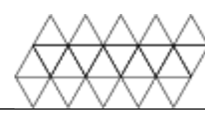
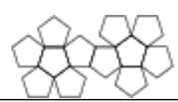
4.2.1 Regular Polygons and Regular Polyhedra

Regular polygons are essential in forming regular polyhedra. After all, polygons are what form the polyhedra themselves. Polygons are flat shapes that are formed by line segments, and do not consist of any depth. Regular polygons refer to flat figures that are equiangular (consists of equal interior angles), and in effect, equal in lengths as well. There are infinitely many polygons. Polyhedra are geometric objects that are formed by connecting polygons with more polygons. At least three faces meet at any given vertex; because of this, there are only a certain number of polyhedra. If any number of faces fewer than three meet at a vertex, the figure would not be closed.



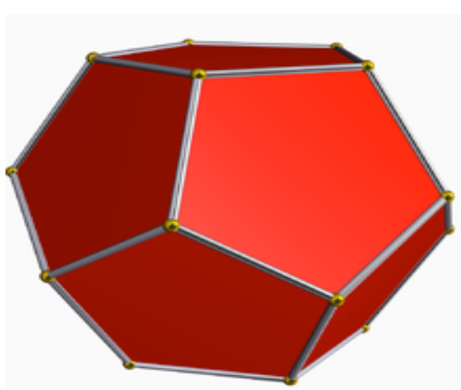
The Platonic solids represent the only regular polyhedra that are possible. The tetrahedron, octahedron, and icosahedron are composed only of equilateral triangles. They differ in the number of faces that meet at each vertex. The cube on the other hand, is composed of squares, and resembles the shape of everyday objects such as an ice cube or dice. The dodecahedron is composed of only regular pentagons.



Regular Polyhedra vs. Regular Polygons	
Platonic Solid	**Unfolded Polyhedron**
Tetrahedron	
Cube	
Octahedron	
Icosahedron	
Dodecahedron	

4.2.2- *Schläfli* symbol

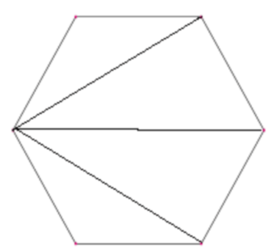
The *Schläfli* symbol is a notation {p,q} used to describe the construction of regular polyhedra (i.e. tetrahedron, cubes, dodecahedron) as well as tiling of the plane. The symbol {p} denotes that the polyhedron is constructed from p-sided regular polygons. For example if p=3, the polyhedron would be constructed from equilateral triangles. The symbol {q} indicates that the polyhedron has q-number of p-gons meeting at each of its vertices. Therefore {p,q} represents the different constructions of polyhedra through the values of {p} and {q}. Therefore {5,3} represents a polyhedron with 3 pentagons around each vertex, otherwise known as a dodecahedron.



Since {p,q} describes characteristics of a polyhedron, we must find their possible values. The values of {p} and {q} are derived from the measurements of the interior angles in a regular polyhedron. To find their possible values we must first look the interior angle theorem:

$(p-2)\pi/p$= measure of an interior angle

This formula allows us to find the measure of each interior angle in a polygon by the separation of a polygon into triangles so that the polygon is tiled completely with triangles. With a hexagon we can see that the 6-gon can be separated in 4 triangles or (6-2) triangles.



We take the amount of triangles within the polygon and multiply by π to find the sum of the measure of the interior angles in the polygon.

$$6-2=4\bullet\pi=720$$

Then divide by the number of sides in the polygon, p.

$$720/6=120°$$

This gives us the measure of each single interior angle in the polygon.

We can now use this theorem using the arbitrary element p where there are (p-2) triangles and p-sides to form the interior angle theorem:

$(p-2)\pi/p=$ measure of an interior angle

This theorem can then be used in a polyhedron rather than a flat polygon to find the sum of interior angles meeting at each vertex by including q, or the number of p-sided polygons at a vertex, in the formula to account for the total number of single interior angles meeting at a vertex into:

$q\bullet(p-2)\pi/p=$sum of interior angles meeting at a vertex

This formula can then be simplified so that the product of this equation remains less than 2π for it refers to the interior angles of a polyhedron which does not lie flat on the plane. Therefore we can create the formula:

$$q\bullet(p-2)\pi/p<2\pi$$

By establishing this formula, we can now test possible values for {p} and {q} for different polygons.

3-gons

$$q\bullet(3-2)\pi/3<2\pi$$

$$q\bullet(1)\pi/3<2\pi$$

$$q\pi/3<2\pi$$

$$q\pi<6\pi$$

$$q<6$$

Through this equation we can conclude that if p=3 the possible values for q= {3,4,5}. The values 1 and 2 do not satisfy q for there cannot be 1 polygon meeting at a vertex in a polyhedron and 2 polygons meeting at a vertex only form 1 edge.

4-gons

$$q\bullet(4-2)\pi/4<2\pi$$

$$q\bullet(2)\pi/4<2\pi$$

$$2q\pi/4<2\pi$$

$$2q\pi<8\pi$$

$$2q<8$$

$$q<4$$

For the value of p=4, the only possible value of q is 3.

5-gons

$q \bullet (5-2)\pi/5 < 2\pi$

$q \bullet (3)\pi/5 < 2\pi$

$3q\pi/5 < 2\pi$

$3q\pi < 10\pi$

$q < 10/3$

From this we conclude that if p=5, then q=3.

6-gons

$q \bullet (6-2)\pi/6 < 2\pi$

$q \bullet (4)\pi/6 < 2\pi$

$4q\pi/6 < 2\pi$

$4q\pi < 12\pi$

$q < 3$

This equation shows that if p=6 then there is no possible value for q due to the fact that q={1,2} can't work.

We can finally conclude that any value of $p>5$ cannot work, for the value of q will not satisfy the characteristics of regular polyhedra. Therefore the only possible *Schläfli* symbols for regular polyhedra are:

$\{3,3\}$

$\{3,4\}$

$\{3,5\}$

$\{4,3\}$

$\{5,3\}$

From these possible values we find that there can only be 5 possible constructions of regular polyhedra for the values of $\{p,q\}$.

4.2.3. Construction of 5 Regular Polyhedra: The Platonic Solids

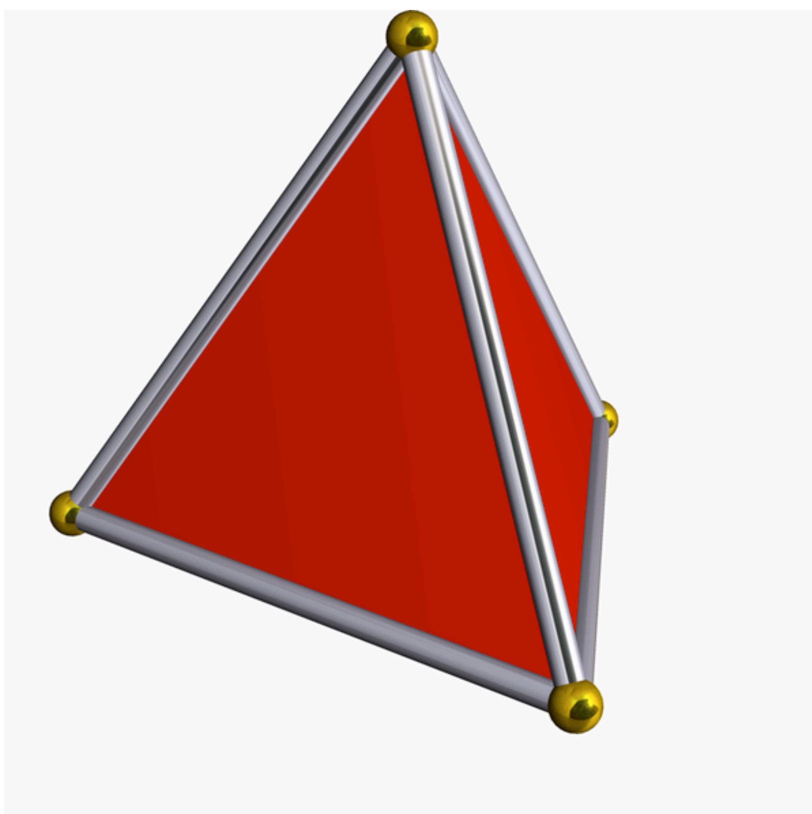
The sum of the interior angles meeting at a vertex must be less than 2π, as shown in the following equation:

$$\mathbf{q\ (p-2)/p < 2\pi}$$

The fact that the sum of the interior angles meeting at a vertex is less than 2π is crucial in that if the sum were to be equal to 2π, then it would be flat on a plane. Moreover, our goal is to create regular polyhedra that need to be 3 dimensional, with the sum of the interior angles being less than 2π. The only possible values for $\{p,q\}$ were therefore $\{3,3\}$, $\{3,4\}$, $\{4,3\}$, $\{3,5\}$ and $\{5,3\}$. The regular polyhedra that satisfy the numerical restrictions possess the *Schläfli* characteristic. In order to construct a regular polyhedron, the faces must be equal and there must be the same number of faces meeting at each vertex.

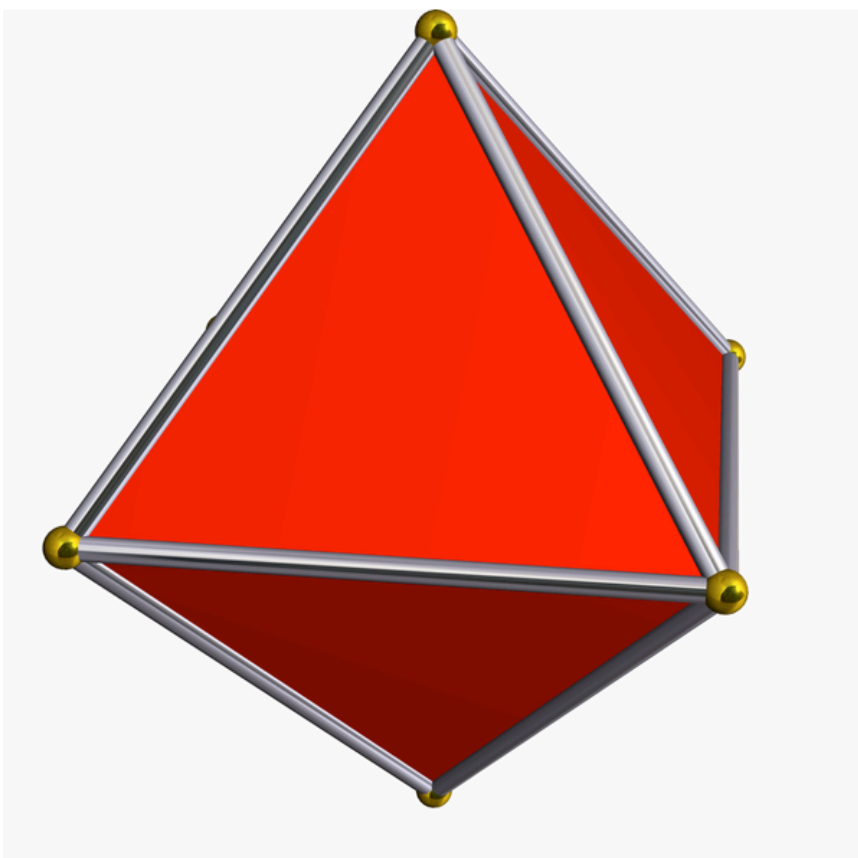
The Tetrahedron

A tetrahedron is composed of four equilateral triangles, three of which meet at one vertex. Three triangles can form a vertex and one more triangle closes the shape, forming a platonic solid with triangular faces. Moreover, this regular convex tetrahedron, can be denoted using the *Schläfli* symbol, {p,q}, as {3,3}. Meaning, 3 3-gons meeting at one vertex, each to form the tetrahedron.



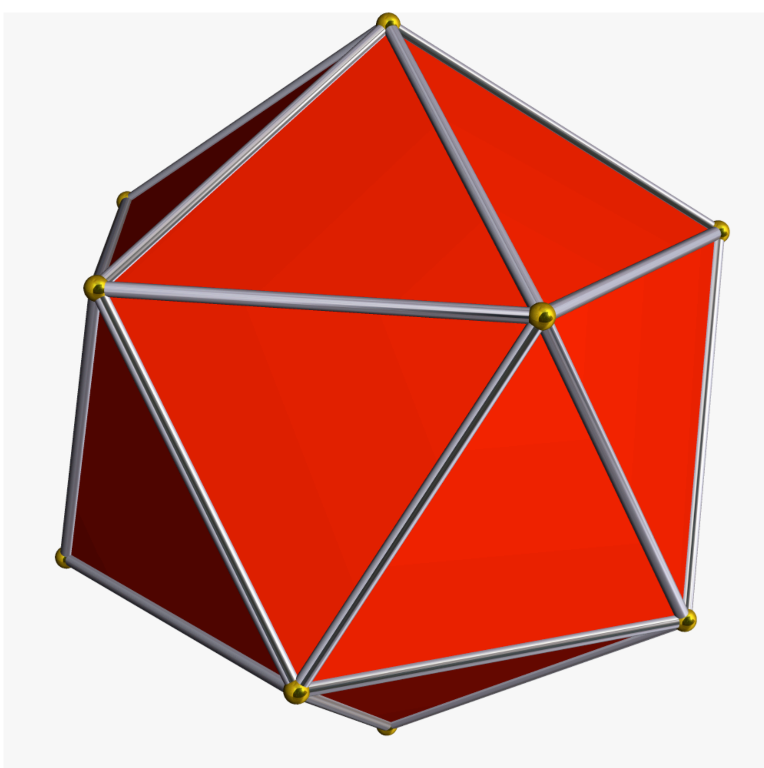
Octahedron

The octahedron, similarly to the tetrahedron, is formed by equilateral triangles, but with four triangles meeting at each vertex. These four triangles form a shape that looks like a pyramid with an open bottom. Moreover, two of these pyramids can connect to form a platonic solid with eight triangular faces. With identical faces from every perspective, this is called a regular octahedron. Using the *Schläfli* symbol, this polyhedron can be expressed as $\{3,4\}$. Meaning, four 3-gons meeting at each vertex.

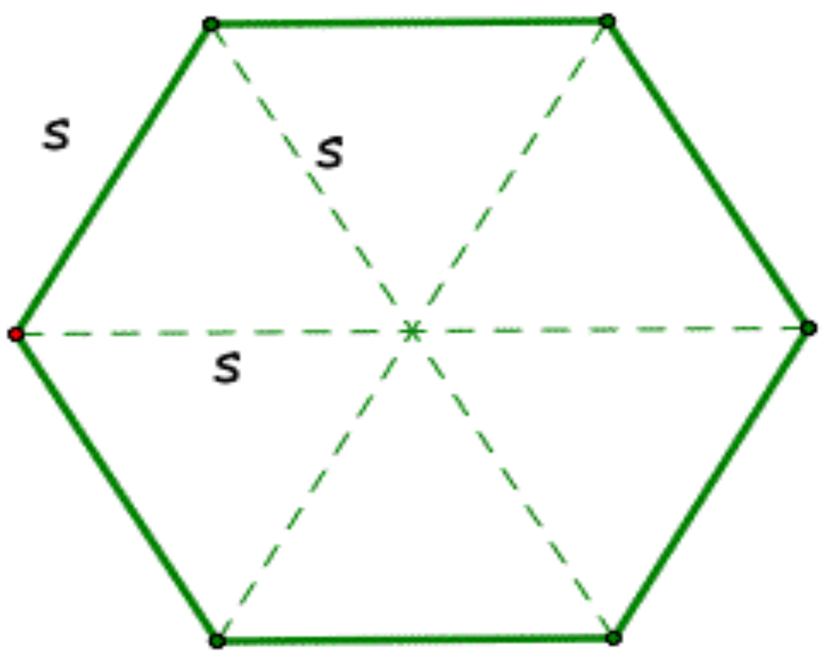


Icosahedron

The last regular polyhedron structured by equilateral triangles is the Icosahedron. In this figure five equilateral triangles converge at one vertex forming a dome-like figure. If two of these figures were coupled together, it would not form a platonic solid, because not all the corners possess five triangles at each vertex. However, to solve this small setback, a ring of ten equilateral triangles can be attached to the center connecting the domes and closing the figure. Ultimately, a platonic solid with twenty faces is formed, the regular icosahedron, represented using the *Schläfli* symbol {3,5}, with five 3-gons meeting at each vertex.

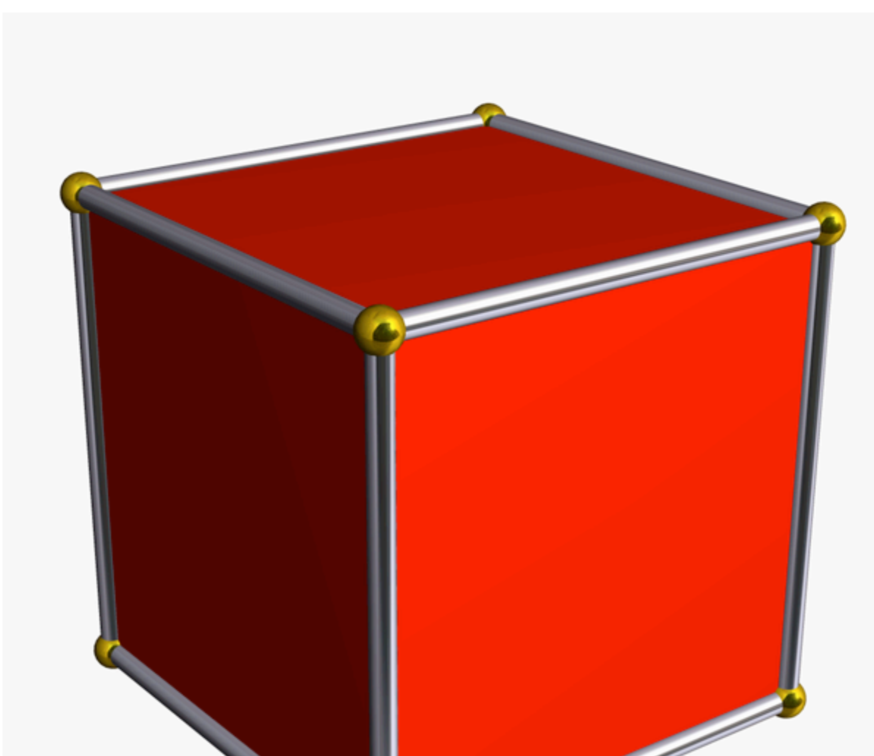


No more regular polyhedra can be formed using equilateral triangles, due to the fact that no more triangles can fit about a common vertex. Consequently, we have now established that there can be only three, four or five of these triangles meeting at one vertex to create regular polyhedra. Moreover, if six equilateral triangles were to meet at a vertex on a plane, then the figure would lie flat on a plane, rendering it a hexagon.



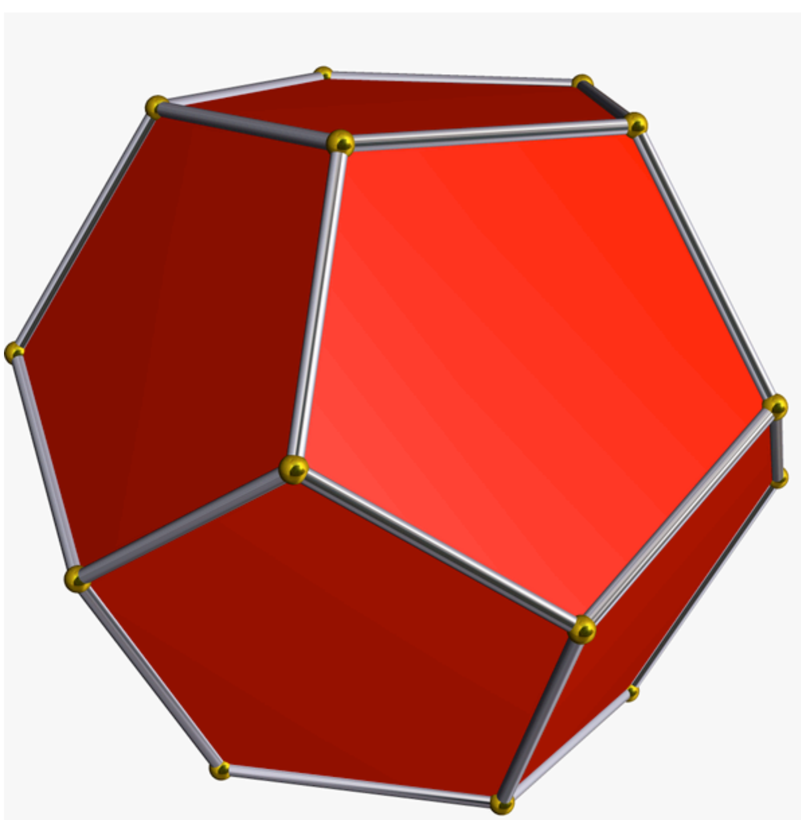
The Cube

The cube can be constructed by one of the simplest regular polygons, the square. Three of these squares can join at a vertex to form one corner; similarly, another three can join to form another corner. These two corners can merge to form a six-sided figure known as the cube. It would be quite impracticable to add another square to one vertex, because four squares meeting at a vertex would lie flat on a plane, like the situation with the hexagon stated previously. The cube is a unique polyhedron, since it is the only one of the five polyhedra composed with square faces. Using the *Schläfli* symbol, the cube is denoted as, {4,3}, or three squares meeting at one vertex.

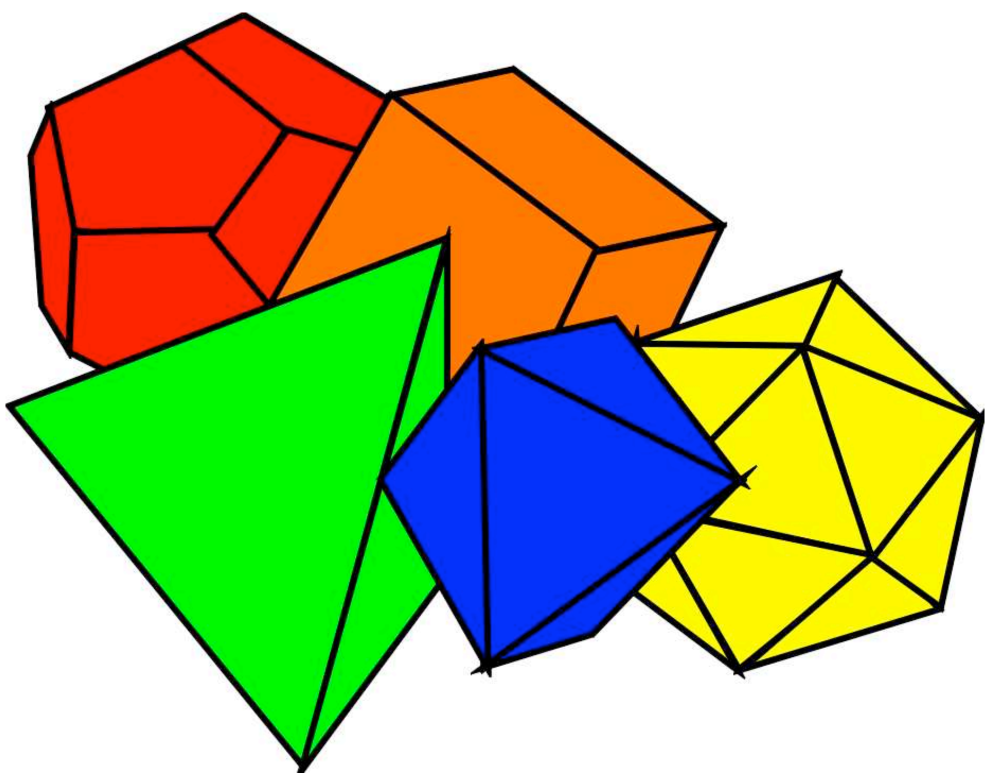


Dodecahedron

Following the simplicity of the square, the next regular polygon we have is the pentagon. Using, three pentagons joined at one vertex, a corner is constructed. Moreover, combining four of these corners, a platonic solid with twelve faces is created, known as the regular dodecahedron. In this case, adding another regular pentagon wouldn't even fit around the common vertex on the plane. Ultimately, the dodecahedron can be represented as {5,3} or three pentagons meeting at each vertex.



These five convex regular polyhedral: the Tetrahedron, Octahedron, Icosahedron, Cube, and Dodecahedron are the only regular polyhedra that can be constructed, that have aesthetically pleasing rotational symmetries. For example, if one were to go on to try the next regular polygon, the hexagon, it would lie flat on the plane. Moreover, regular polygons with more than six sides would fail to fit around a common vertex completely. Therefore, there are only five platonic solids that can be constructed and comply with the numerical restrictions of interior angles being less than 2π, $p \geq 3$ and $p \leq 6$. Although it was difficult to reach this conclusion to the phenomenon of Platonic solids, we have to now find out the properties of these solids and how they hold true to each one of them.

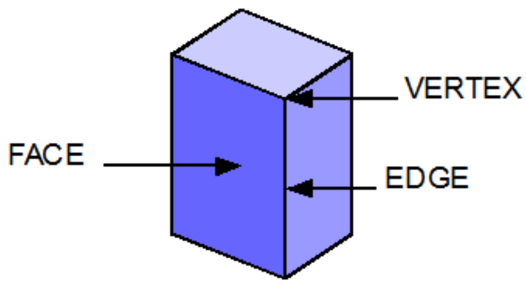


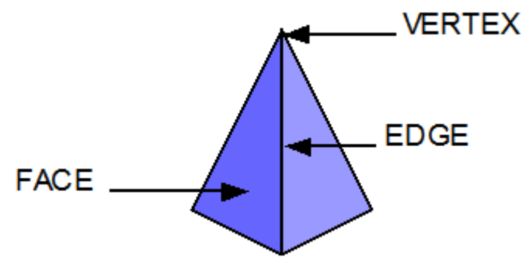
4.2.4 Vertices, edges, and faces

Previously, we constructed five regular polyhedra. These polyhedra were the tetrahedron, the cube, the octahedron, the dodecahedron, and the icosahedron. To be a polyhedron, the number of q faces that meet at a vertex in a p-gon have to satisfy the equation:

The only possible values for {p,q} that satisfy the equation are {3,3} of a tetrehedron, {3,4} of a octahedron, {3,5} of a icosahedron, {4,3} of a cube, and {5,3} of a dodecahedron. The value for P could not be less than two because polygons consist of 3 or more faces but when p exceeded 5, q would result in a fraction so anything greater than 5 and less than 2 was not possible. Thus the five platonic solids were created.

Then we studied the geometric elements of those five platonic solids. These elements include vertices, edges, and faces which are found on polygons to make up three- dimensional objects. A vertex is a point where three or more planes intersect. An edge is the intersection of two faces. A face is a section of a plane.





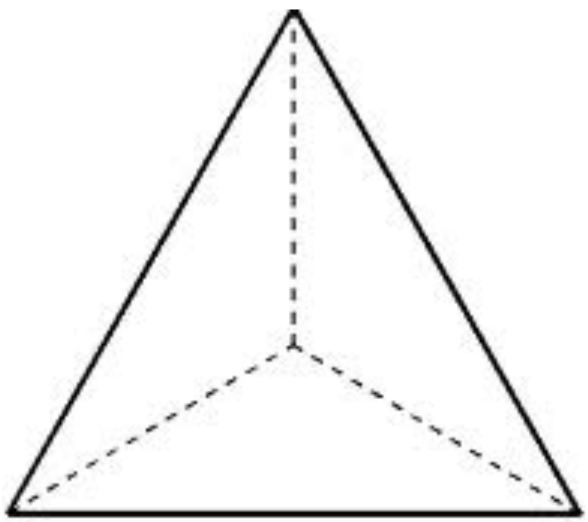
Then we physically counted the number of vertices, edges, and faces on the five platonic solids.

Platonic Solids	Vertices (V)	Edges (E)	Faces(F)	V-E+F	p	q
Tetrahedron	**4**	**6**	**4**	**2**	**3**	**3**
Octahedron	**6**	**12**	**8**	**2**	**3**	**4**
Icosahedron	**12**	**30**	**20**	**2**	**3**	**5**
Cube	**8**	**12**	**6**	**2**	**4**	**3**
Dodecahedron	**20**	**30**	**12**	**2**	**5**	**3**

The results lead us further into investigating duals of the platonic solids as well as uniqueness of the Euler characteristic and the counting arguments.

4.2.5. Uniqueness of Platonic Solids – Counting Arguments and Euler Characteristic

As said before, there are only a set amount of platonic solids that can be formed: five in total which are the Tetrahedron, Octahedron, Cube, Icosahedron, and the Dodecahedron, according to the Schläfli symbol $\{p, q\}$. Using counting arguments and the Euler Characteristic, one can find its vertices, edges, and faces and see that those values for V, E, and F are fixed for each polyhedron.



A Tetrahedron with Schläfli symbol {3, 3} has 3 triangles meeting at a vertex.

Counting Arguments:

Each face has p vertices so we can put the equation as pF which represents the number of vertices of all polygons before the faces are glued together. Therefore, we can write the amount of vertices of a regular polyhedron as since each of the vertices shares q faces after we glue the shapes to the polyhedron. pF also represents the number of edges of all polygons before the edges are glued together. We can also write the amount of edges after gluing as   because each of the faces shares an edge with another face. Each edge shares 2 faces after gluing the shapes. So another way to write this is:

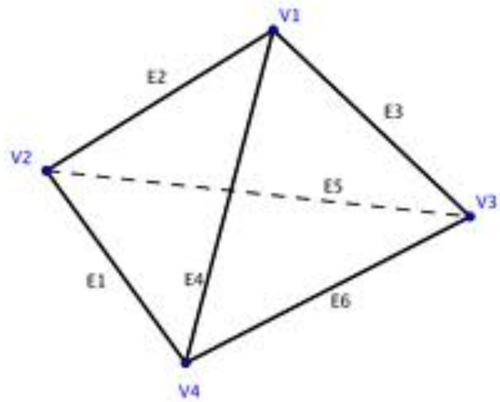
E = or 2E = pF

= V or qV = pF

This implies:

2E = qV = pF

For example, let's take a tetrahedron which has Schläfli symbol {3, 3}. A tetrahedron before the faces are glued has 12 vertices, 12 edges, and 4 faces. When its faces are glued, a tetrahedron has 4 vertices, 6 edges, and 4 faces.



A tetrahedron with 4 vertices, 6 edges and 4 faces

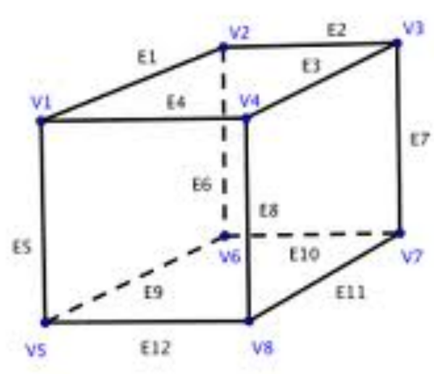
This works with the formulas above when you plug in p with 3, F with 4, and q with 3.

2E = pF would be 2 x 6 = 3 x 4 ➔ 12 = 12 ✓

qV = pF would be 3 x 4 = 3 x 4 ➔ 12 = 12 ✓

2E = qV = pF = 12 ✓ 🡐 As we can see, it fits with the formula.

Another example we can illustrate that this fits is a cube which has Schläfli symbol {4, 3}. A cube before the faces are glued has 24 vertices, 24 edges, and 6 faces. When its faces are glued, a cube has 8 vertices, 12 edges, and 6 faces.



A cube with Schläfli symbol {4, 3} has 3 squares meeting at a vertex.

This works with the formulas above when you plug in p with 4, F with 6, and q with 3.

2E = pF would be 2 x 12 = 4 x 6 ➔ 24 = 24 ✓

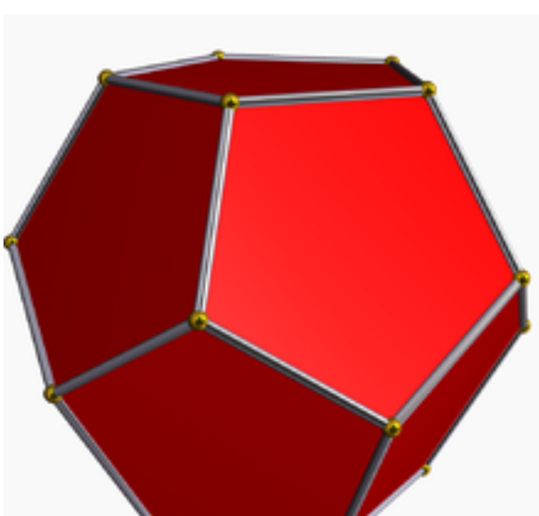
qV = pF would be 3 x 8 = 4 x 6 ➔ 12 = 12 ✓

2E = qV = pF = 24 ← We can see that the formula works with a cube, too.

2E = qV= pF holds for all 5 regular polyhedrons.

Euler Characteristic:

The Euler Equation or Characteristic states that V – E + F = 2 in all cases for polyhedrons. Let **R** equal a regular polyhedron which has V vertices, E edges, and F faces. Assume the polyhedron's faces are laid out on a flat surface. Using the Schläfli symbol $\{p, q\}$, p represents the type of regular polygons in a polyhedron and q represents how many of these m-gons meet at each vertex.



A Dodecahedron with Schläfli symbol {5, 3} has 3 pentagons meeting at a vertex.

We now know that pF = 2E = qV. We can get V, F, and E values if we plug in the counting arguments and p and q values for each polyhedron in the Euler's characteristic. If p and q are fixed, we would know that there are only fixed values for V, E, and F which shows the uniqueness of a polyhedron.

We can see it to find the vertices and edges for a Tetrahedron {3, 3}.

Using Euler's characteristic V – E + F = 2, we can use the above values and plug them in for V and E.

$$F = 4 \checkmark$$

The same goes for finding the vertices, faces, and edges of a Cube {4, 3}:

Plugging in to V – E + F = 2:

F = 6 ✓

For an Octahedron {3, 4}:

Plugging in to V – E + F = 2:

F = 8 ✓

For a Dodecahedron {5, 3}:

Plugging in to V – E + F = 2:

F = 12 ✓

For an Icosahedron {3, 5}:

Plugging in to V – E + F = 2:

$$F = 20 \checkmark$$

4.2.6 Duals of Platonic solids

Duality is the property in a pair of polyhedra where one polyhedron can be inscribed in another polyhedron if one had its vertices and faces interchanged. For example:

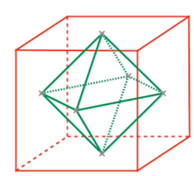


Figure 1 – Cube and Octahedron

If a polyhedron has vertices in the center of the faces of a cube (Figure a), the vertices would form an octahedron (Figure b).

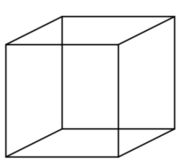


Figure a – Cube

Conversely, if vertices were in the centers of the faces of an octahedron, they would form a cube. The cube and the octahedron are duals. The duality is visualized in Figure 1.

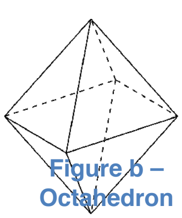


Figure b – Octahedron

Referring back to the Schläfli symbol, the notation for polyhedra is {p,q} where p is type p-gon and q is the amount of p-gons meeting at a vertex. The cube is a {4,3} polyhedron and the octahedron is a {3,4} polyhedron. One can notice that the p and q values of a polyhedron are interchanged in its dual. If {p*,q*} is the dual of a polyhedron, p*=q and q*=p. It can also be noted that (p*)*=p.

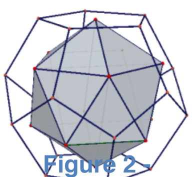


Figure 2 – Icosahedron and Dodecahedron

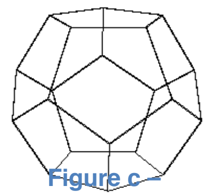


Figure c – Dodecahedron

The dodecahedron (Figure c) and the icosahedron (Figure d) is another pair of polyhedra. Duality can be seen again using the same procedure. If one were to create vertices in the center of the faces of the icosahedron, the resulting shape is a Dodecahedron.

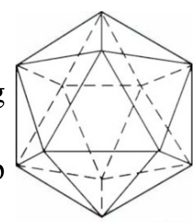


Figure d – Icosahedron

The same observation can be made with the icosahedron when finding its dual. Like the previous example, the Schläfli symbol has interchanging p and q values between the duals. The dodecahedron is {5,3} and the icosahedron has {3,5}.

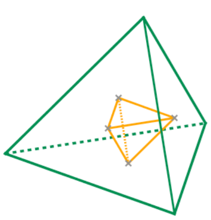


Figure 3 - Tetrahedrons

A tetrahedron is dual to itself as seen in Figure 3. Tetrahedrons have the same number sides as the number of polygons connected at a vertex. The dual of a {3,3} polyhedron will remain as {3,3}.

These three examples are all of the duals among platonic solids.

4.2.7 A related Question: Regular Tilings of the Plane.

"Tiling the plane" means to cover a plane with non-overlapping regular p-gons so that adjacent polygons form an edge and the intersection of two edges make a vertex. The result is the same number of polygons meeting at each vertex. The number of polygons that meet at a vertex is represented by q similarly to the q value in the study of platonic solids. For example:

One can tile the plane with squares (Figure 4). The square is a 4-gon and each vertex is the intersection of 4 squares. Under the Schläfli symbol, this is a (4,4)-tiling.

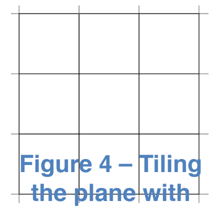


Figure 4 – Tiling the plane with squares

Similarly to how there is a finite number of platonic solids, there is also a finite number of (p,q)-tilings. There are numerical restrictions for the type of p-gons and the amount of p-gons connected at a vertex. To find all types of (p,q)-tilings:

1) The interior angle of a regular p-gon can be calculated by $((p-2)\pi)/p$. A square is a p-gon so its interior angle is equal to $((4-2)\pi)/4$. The square's interior angle is equal to $\pi/2$.

2) The sum of the interior angles of polygons meeting at a vertex must be 2π. In the case of the square, its interior angles are all $\pi/2$. There are four squares meeting a vertex and therefore the sum of the interior angles is 2π. Combining with the previous statement, we can discover all valid (p,q)-tiling by computing $2 = (q(p-2))/p$. If the values of p and do not result as integers for any p or q, tiling is not possible for that example.

3) A 5-gon cannot be tiled without overlapping. This can be seen algebraically when finding its interior angle with the same formula as above. The number of polygons meeting at a vertex (q) would be 10/3. Tiling is not possible.

4) The minimum amount of polygons that can intersect at a vertex that can be tiled is 3. If limited to 2, it would only possible to create a line. If it limited to one, it would only be the vertex itself.

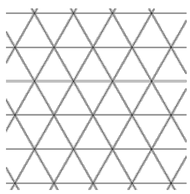


Figure 5 – A tiling of triangles (3,6)

With these statements, one can conclude that the only possible regular p-gon tilings possible are (3,6),(4,4), and (6,3).

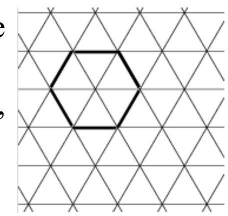


Figure 7 – A hexagon in a tiling of triangles.

When comparing the tiling of the plane to platonic solids, duality can also be found in both examples. As shown in 4.2.6, the cube {4,3} and the octahedron {3,4} are duals.

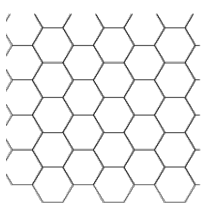


Figure 6 – A tiling of hexagons (6,3)

A similar example can be seen when comparing regular triangle (3,6) and regular hexagon (6,3) tilings (Figures 5 and 6). The p and q values interchange in the Schläfli symbol. On a plane, the triangle and the hexagon are dual to each other Figure 7).

This is also seen in a square where it is dual to itself (Figure 6). Duality is a phenomenon found in both tiling and polyhedrons.

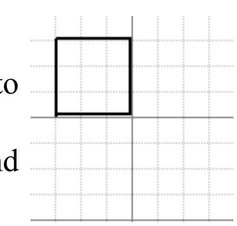


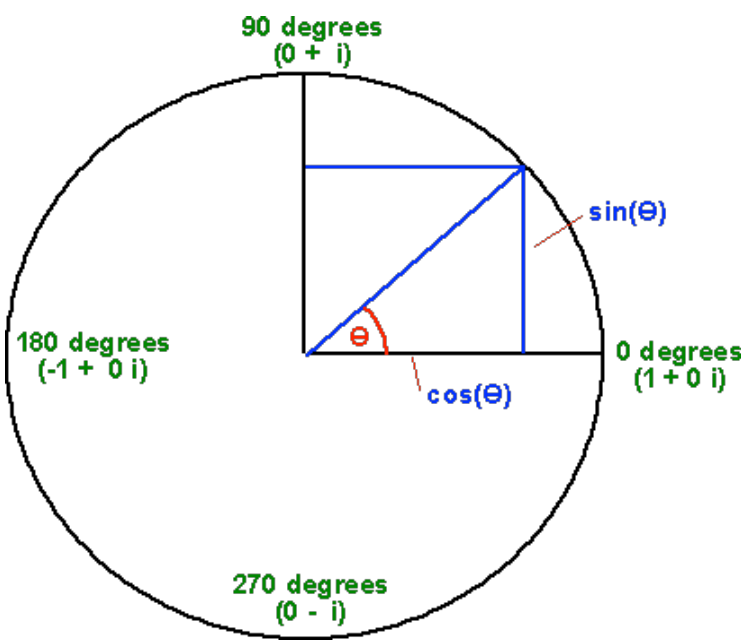
Figure 8 – A square in a tiling of squares

4.3 Symmetries of the Platonic Solids

The platonic solids (regular polyhedra) have multiple rotational symmetries, but we will only be focusing on three of them: the Tetrahedra, Cube, and Dodecahedron. Since the duals of the Cube and Dodecahedron, the Octahedron and Icosahedron (respectively), have the same total number of symmetries as the originals, it is unnecessary to focus on them. The originals are represented as Rot (P), while the duals are represented by Rot (P*), where P is a polyhedron. Interestingly enough, the geometry (rotational groups) can be encoded in algebraic terms (permutation groups), which shows a true connection between the two major math subjects.

4.3.1. Three-Dimensional Rotations

In two-dimensional rotation, we are not faced with the problem of clockwise and counterclockwise rotations. This is because the rotations of two-dimensional figures are based on the origin, or center of the figures (as displayed in the figure below).



Three-dimensional rotation is the rotation of two points on different axes (often defined as the z-axis) on the polyhedron. The rotation of (x, θ) (where x is the axis of symmetry and θ is the angle of rotation) is also the equivalent of $(-x, -\theta)$. θ is rotation counterclockwise, where $-\theta$ is rotation clockwise. Simply put, every time you rotate a point, x, by θ, then the opposite point, -x, is rotated by $-\theta$. Rotating –x by $-\theta$ would preserve and send the same points on the sphere to the same new points as rotating x by θ would. The inverse of (x, θ) would be $(x, -\theta)$. Three-dimensional rotation will involve the axes between opposite vertices, centers of opposite faces, and centers of opposite edges. We will be focusing on rotational symmetries, symmetries that preserve the shape, but send certain vertices, faces, and edges to new ones.

4.3.2. Counting Rotational Symmetries

We counted the rotational the symmetries of the tetrahedron is 12, which equals 4(3), 6 (2), and 3(3). The total number of rotational symmetries of (p,q), $|Rot\{p,q\}|$, is equal to pF, qV, and 2E.

$$|Rot\{p,q\}| = pF = qV = 2E$$

Let's prove this. We already know V-E+F=2. Now, we shall do a naïve count of pF, qV, and 2E. For pF, we look at the Rot (c, multiple of $2\pi/p$), where c is the center of a face. Naively, we would not consider pF to be a multiple of $2\pi/p$. Next, we look at Rot (v, multiple of $2\pi/q$). Naively, we consider qV to be such rotations. With 2E to add into the mix, we would have a naïve sum of pF+qV+2E. However, we over counted because of the identities (e), therefore we must add the identity total, 1, to the naïve sum, but subtract F, V, and E from it.

$$1 + pF + qV + 2E - (F + V + E)$$

As previously mentioned, Rot (x, θ) is equivalent to Rot (-x, -θ). Since this is the case, we double counted, therefore we must divide the above variables by 2.

$$1 + \tfrac{1}{2} (pF + qV + 2E - (F + V + E))$$

$$1 + \tfrac{1}{2} (pF + qV - (-E + F + V))$$

As previously mentioned, V-E+F=2, so:

$$1 + \tfrac{1}{2} (pF + qV - 2)$$

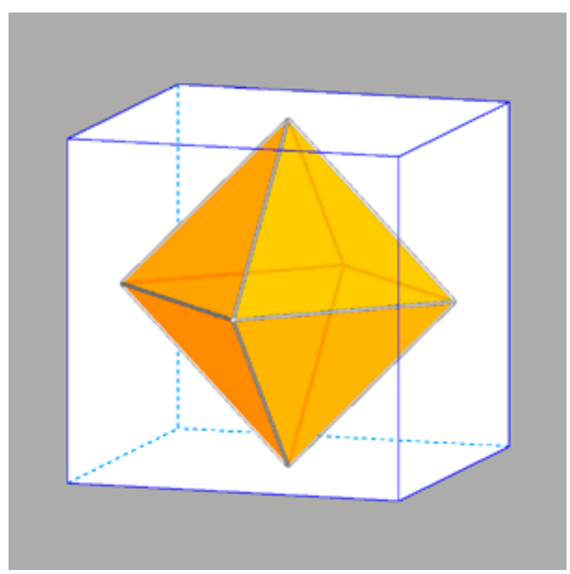
Since pf=qV, we will replace qV with pF:

$$1 + \tfrac{1}{2} (2pF - 2)$$

The result will be pF.

4.3.3 Rot(P) and Rot(P*)

The order of the rotations of a polyhedron always equals that of its dual. This is due to the fact that a shapes dual is formed by connecting the center of the faces to form a polyhedron. When this occurs, the number of faces and the number of vertices always swaps. So, in general, the difference between a



Octahedron Inscribed In a Cube (Its Dual) polyhedron and its dual is that the symmetries of the rotations about centers of opposite faces switches with the symmetries of the rotations about opposite pairs of vertices. The rotations about the edges always are the same between the two. This really shows the beauty of polyhedra and their duals.For example: a cube has 6 faces, 12 edges, and 8 vertices. Its dual, a octahedron has 8 faces, 12 edges, and 6 vertices.

4.3.4 Groups Properties- the Problem with Closure

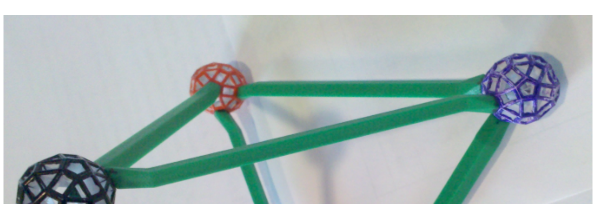
Group theory is a section of abstract algebra that studies the algebraic structure of groups. Groups are concrete in the sense that there are rigid qualifications to be a group. If one property is missing, then a set is not a group. These properties include: closure, associativity, the identity, and the inverse. The geometry that we have been studying for the past six weeks corresponds with abstract algebra in the sense that each Platonic solid carries these four group properties. To show these properties algebraically is less difficult than to show these properties visually.

Associativity is naturally implied because the operation of composition is associative. We used composition of functions to manipulate the platonic solids to test the other four properties. It is true that rotation $\pi/2$ composed with a rotation of $3\pi/2$ is the same rotation as $3\pi/2$ composed with $\pi/2$ on the same axis. This shows that composition of $(1\bullet 2)\bullet 3 = 1\bullet(2\bullet 3)$

The Identity is present because a rotation of 0° doesn't move the solid at all. The solid is in the same position as before. A rotation of 2π also rotates the shape 360°, which is also putting the shape back into its original position.

The inverse is always present because it is the negation of the rotation given.

$$\mathrm{Rot}\,(\theta)^{-1} = \mathrm{Rot}(-\theta)$$

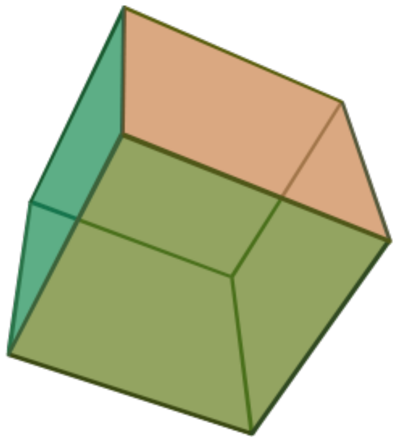


$\mathrm{Rot}\,(\pi/2)^{-1} = \mathrm{Rot}\,(3\pi/2)$ because it is simply rotation of the same angle, just in the opposite direction.

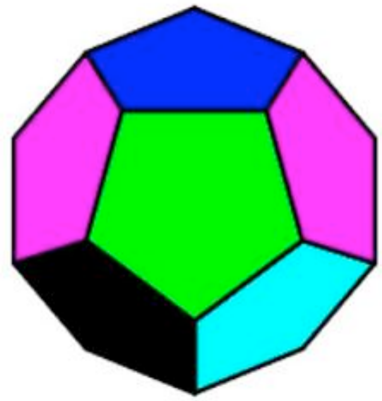
The problem we are faced with is to fully prove that the Platonic solids are groups themselves and have closure. Closure would be taking any two elements in the set, performing the given operation, and resulting with an element that is also inside of the set. The problem is that there are many axes to be accounted for.

The tetrahedron, for example, has four vertices, which can be rotated individually.

When one vertex is fixed, is it easy to keep track of the rotations. It is when the tetrahedron is rotated about different axes that we lose the orientation visually. To help with this problem, we labeled each vertex with a different color or as (1,2,3,4) to keep track of the vertices as they rotated. The tetrahedron was manageable because there are only four vertices. Once we constructed the cube and dodecahedron, physically keeping up with the rotation about different axes was nearly impossible. This obstruction has inspired us to look at the algebra behind rotations: permutations.



Cube



Dodecahedron

Instead of physically seeing that when rotating about one vertex, the others where sent to different ones, we expressed it through the decomposition of functions.

If vertex 1 was fixed, Rot ($2\pi/3$) would send vertex 2 $\Rightarrow$ 3, 3 $\Rightarrow$ 4, and 4 $\Rightarrow$ 2. Algebraically, this can be written as: (234) (1 is fixed so it is sent to itself).

A different rotation would have a different sequence because 2 may be sent to 4 this time, instead of vertex 3.

As you can see, things can get messy.

We determined the finite number of permutations of each polyhedra by discovering each symmetry and recording how many times each symmetry occurred.

For the tetrahedron, fixing one vertex would give two rotations. Since there are eight vertices, there are eight symmetries of $2\pi/3$ and $4\pi/3$.

When we looked at the opposite pairs of edges, we discovered that rotating by π would also give us symmetry. Since there are three pairs of opposite edges, there are three π symmetries. Finally, we have the identity symmetry, e.

In total, that's 12 symmetries. To record these, we simply tracked a few visually then saw a pattern.

While fixing vertex one, the remaining vertices could be sent up to two places. 2 $\Rightarrow$3,4. 3$\Rightarrow$4,2. 4$\Rightarrow$2,3. The possibilities aren't endless, but there are many to keep track of.

Instead of listing them out, we noticed that the options for the first number in the permutation is either 1,2,3, or 4. The second number could only be one of three remaining options, and the third number had only two options. This was a counting argument, to keep track of our data.

Since the rotations about vertices have an order of three, there are only three digit potions available for four numbers.

$4*3*2 = 24$. 24 possibilities.

This is impossible because there are only 12 symmetries. More numbers were accounted for than they should have been, so when we divided the product by the order, we got the correct number of permutation possibilities.

$24/3=8$. 8 different rotation possibilities of order 3.

(123), (124), (134), (132), (143), (142), (234), (243).

Now we can to find a way to record our rotations about opposite edges by π.

This rotation has an order of two, so that leaves two digit positions. But, we are working with pairs, so we will need two of these two digit positions, resulting in four positions.

$(4*3)\ (2*1) = 24$. 24 possibilities.

Once again, we have over-counted. We must account for

$(ab)(cd) = (ba)(dc) = (cd)(ab)$

$$2 \quad * \quad 2 \quad * \quad 2$$

$24/8 = 3$. 3 possible (2x2) permutations.

(12)(34), (13)(24), (14)(23).

And we have the identity, e.

Listing was the hard part. To show closure properly, we must compose 3-cycles with 3 cycles, 3 cycles with (2x2) cycles, and all other combinations and get an element that is still inside of our potential group.

To successfully express the end product of the permutations, one must know how to manipulate permutations algebraically.

Like before, the vertices are being sent to different place and that is expressed with their labeled number. If we were speaking generally and not specifically for each vertex, then the only way to know the end result would be to know how to do the algebra.

In the composition, we want to know what happens when vertex 1 is sent to vertex 2 is composed with vertex 1 going to vertex 3.

We must read and perform cycle decomposition from right to left.
We must also but the smallest number in the permutation first. The numerical order after the smallest number irrelevant.

It is written as (13) • (12) because we want to know what happens when we do (12) then (13). In (12) one is being sent to 2, then we want to follow were 2 is being sent to because that end result will be 1's end result because it initially went there in the first place. In (13), 2 is being sent to itself if is absent from the parenthesis. So that means that 1 remains being sent to 2. We write that as (12…
But the composition is not finished. We still need to see where 2 and 3 get sent.

We still being right to left for each number we are tracking. On the right, (12), we see that 2 goes back to 1. Then on the left, we see that 1 goes straight to 3. So that means 2 gets sent to 3. We add 3 to our open composition. (123…

Now we want to finally track where 3 end up.

Starting on the right, we see that 3 is absent so it gets sent to itself. On the left, (13), 3 gets sent back to one. Since 1 is on the beginning of our cycle, we can safely close the composition because every point has a place to be sent to.

The result of two 2-cycles composed with each other is a 3-cycle.

(123) • (142) = (143).

(143) is one of eight rotations of order 3.

(14)(23) • (134) = (123).

(123) is another rotation of order 3.

This successfully proves that all elements contained on the tetrahedron can be composed with each other and generate a different symmetry inside of the group.

Proving that closure exists within a cube and dodecahedron is more difficult because there are more numbers to work with, but it can be done.

Permutations are 1:1. For every input, there is only one output. In other words, the function is sent to itself.

Distinct elements of (1,2,3,4) are mapped to other distinct elements of (1,2,3,4).

If σ is a permutation, then σ is 1:1. For any k, k= 1,2,3 or 4; there is some i = 1,2,3 or 4 such that $\sigma(i)=k$

$\sigma(i_1)=k$, $\sigma(i_2)=k$, then $\sigma(i_1)= \sigma(i_2)$.

$i_1= i_2$? i_1 is unique. i_2 is unique. $i_1 \neq i_2$

This shows that each permutation is (1:1) that is there is only one place it can be sent.

4.3.5 The Symmetric and Alternating Groups

Now that closure has been shown for the composition of elements of the tetrahedron, we can show that any set of unique permutations can be classified as a group.

The Set of all permutations of {1,2,3,4} is a group called; the Symmetric Group of 4 elements denoted S_4. S_4 is all permutations of {1,2,3,4}.

The order of S_4 is 4! $4*3*2*1$ or 24.

We want to prove that S_4 is a group under composition to show that any set of permutations is also a group.

Closure:

If σ_1, σ_2 are permutations, the so is $\sigma_1 \bullet \sigma_2$.

Need to show: If $i \neq j$ then $\sigma_1 \bullet \sigma_2$ (i) $\neq \sigma_1 \bullet \sigma_2$. (j)

$\sigma_1 = i$, $\sigma_2 = j$

$i \neq j$

$\sigma_1 \neq \sigma_2$.

$\sigma(i)=k$, $\sigma(j)=k$, then $\sigma(i)= \sigma(j)$ $\quad i \neq j$

$\sigma_1(\sigma_2(i)) \neq \sigma_1(\sigma_2(j))$

Associativity: Given by composition of functions

Identity: Given

Inverses:

The Inverses of a function must send the original function back to itself.

Rule: if $k= \sigma$ (i) then $\sigma^{-1}(k) = i$

Show that σ^{-1} is a permutation such that:

$\sigma \bullet \sigma^{-1} = \sigma^{-1} \bullet \sigma = e$

Rule:

$\sigma^{-1}(k) = i$, where i is a unique element of $\{1,2,3,4\}$ such that $\sigma(i)=k$

$\sigma^{-1}(k) = i$

$\sigma(\sigma^{-1}(i)) = i$

$k_1 \neq k_2 \;=\; \sigma(i_1) \neq \sigma(i_2)$

$i_1 = \sigma^{-1}(k_1)$, $i_2 = \sigma^{-1}(k_2)$

Suppose :

$\sigma^{-1}(k_1) = \sigma^{-1}(k_2)$. Now multiply each side by σ.

$\sigma(\sigma^{-1}(k_1)) = (\sigma^{-1}(k_2))\sigma$

$k_1 = k_2$

$k = \sigma(i)$ then $\sigma^{-1}(k) = i$

(When sigma does i, it gets sent to k. when sigma goes under the inverse, it has to be sent back to its original, i. that means that k's inverse must be i because it gets sent back to its original.)

S_n = all permutations of $\{1,2,3,4,\ldots,n\}$

The order of S_n is n!

Cycle decomposition of permutations is an easy, algebraic way to show where all elements get sent to.

We used cycle decomposition to more easily map where vertices of a tetrahedron go when permuted.

The alternating group is a subgroup of the symmetric group. (A_n is a subgroup of S_n)

Alternating groups closely examine the rotations and symmetries of the polyhedra specifically. The alternating groups contain only an Even number of 2-cycles. The reason for this is that closure could be shown and that all alternating groups can be subgroups to symmetric groups of a higher value.

Example: Turning S_4 into its alternating group, A_4

1-cycles: (e)

2-cycles: (12), (13), (14), (23), (24), (34)

3-cycles: (123), (124), (134), (132), (143), (142), (234), (243)

4-cycles: (1234), (1243), (1324), (1342), (1423), (1432)

2x2- cycles: (12)(34), (13)(24), (14)(23)

Alternating groups can only contain and even amount of 2-cycles.

Zero 2-cycles: (the identity)

Two 2-cycles: (the 2x2 and the 3-cycles)

3-cycles are two 2-cycles composed together

Example: $(13)\cdot(14) = (143)$

(12) is an example of an odd number of 2-cycles because it is only one 2-cycle. Itself.

Explicitly, A_4 is:

1-cycles: (e)

3-cycles: (123), (124), (134), (132), (143), (142), (234), (243)

2x2- cycles: (12)(34), (13)(24), (14)(23)

The identity, the rotations of $(2\pi/3)$ and $(4\pi/3)$ for the 4 different vertices, and the rotations of π about the 3 pairs of centers of opposite edges.

A_n is all the even permutations of S_n.

In general, $|A_n| = |S_n|/2$.

$|S_4|= 24$. $24/2= 12$. $A_4=12$.

We will soon see that A_4 "=" the symmetries of the Tetrahedron.

Another important Symmetric Group is A_5.

S_5 contains all of the permutations of $5! =120$

The alternating group A_5 is more significant to us than S_5 because it corresponds with all of the rotations of a dodecahedron.

A_5 is a subgroup of S_5. A_4 and A_3 are also contained in A_5.

A_5 contains permutations of order 5,4,3, and 2. These permutations correspond with the rotation of the dodecahedron, similarly to A_4 and the tetrahedron.

A_5 is a subgroup of S_5 which contains all even and odd permutations.

Some examples of elements inside of S_5 are:

5-cycles: (12345), (15432), (12435)

4-cycles: (1235), (1523), (1452)

2x2-cycles: (12)(45), (15)(34)

2x3-cycles: (123)(45), (145)(23)

3-cycles: (345), (245), (145)

2-cycles: (12), (45), (25)

We care about the even permutations because we consider them isomorphic to the rotations of the dodecahedron. And that is a group because they show closure, associativity, the identity and the inverse.

A_5 “=” the symmetries of a dodecahedron

Recording these permutations is important because it is hard to visualize rotating a 20-vertex figure. We track the rotations about different vertices, edges and faces by using cycle decomposition.

For example: a rotation about the faces by $2\pi/5$ and a rotation about the center of edges by π

$(14)(25) \circ (13254) = (135)$

(135) is a rotation about the center of faces or a 3-cycle, which is contained in A_5.

What is special about alternating groups is that since A_4 is a subgroup of A_5, it is possible to possible to fit a tetrahedron inside of a dodecahedron. Since there are 12 elements in A_4 and 60 in A_5, it is possible to fit 5 (60/12=5) tetrahedron inside of a dodecahedron.

The symmetric groups are important but the Alternating groups are more significant to us because they help express difficult visualizations with simple algebra.

Each alternating group of a lower order is contained in an alternating group of a higher order which is why we call them subgroups. The alternating groups hold the four properties of being a group while demonstrating the algebraic way to explore the Platonic Solids.

4.3.6. The Tetrahedron and A_4

Vertices: 4

Edges: 6

Faces: 4

Like stated before, the tetrahedron is a Platonic solid that has a set number of rotational symmetries.

-When one vertex is rotated by $2\pi/3$, the tetrahedron looks as if it was never rotated, but it was. The order of the vertices is different, if labeled. This is similar to if the tetrahedron is being rotated by $4\pi/3$. And if the tetrahedron was rotated once more by $2\pi/3$, then it would return to its original orientation. $\text{Rot}(2\pi/3) \bullet \text{Rot}(2\pi/3) \bullet \text{Rot}(2\pi/3) = \text{Rot}(2\pi) = \text{Rot}(0)$

That is also how we came about the identity, e.

If those steps are repeated for each of the four vertices, then we would get a total of $3 * 4$ symmetries. The problem is that we counted the identity four times when it only needs to be counted once for all cases of symmetries. So with removing the identity, we have $4 * 2$ symmetries.

8 total symmetries for rotating the tetrahedron a multiple of $2\pi/3$ about any vertex.

When the tetrahedron is rotated about its opposite edges, is also creates a symmetry.

A rotation of π makes the tetrahedron look as if it was never rotated, but it was. The vertices are not in the same orientation. If rotated by π once more, it goes back to its original orientation, the identity.

Since there are 6 edges, that's means there are 3 pairs of edges and 1 rotation $*$ the 3 pairs makes 3 symmetries in total.

Now to count:

$8+3+1 = 12$.

12 Symmetries of the tetrahedron.

To record these symmetries in cycle decomposition would be the same as denoting A_4.

The rotation of $2\pi/3$ is considered a 3-cycle because it takes 3 of these rotations to generate the identity. This 3-cycle is denoted:

(abc) in cycle decomposition.

When one vertex is fixed, it means that is never changes its position. If vertex 1 was fixed, the permutations of the other 2 vertices would look like (234) or (243).

The order of numbers inside the permutation depends on the angle of rotation. In a rotation of $2\pi/3$, $2\Rightarrow3$, $3\Rightarrow4$ and $4\Rightarrow2$. So that is expressed as (234). If the tetrahedron was rotated about vertex 1 by $4\pi/3$, $2\Rightarrow4$, $3\Rightarrow2$, and $4\Rightarrow3$. That is expressed as (243).

All permutations of the tetrahedron:

Vertices:

Fixed: 1- (234), (243)

2- (134), (143)

3- (124), (142)

4- (123), (132)

Opposite Edges:

(12)(34)

(13)(24)

(14)(23)

The Identity: (e)

12 symmetries.

Notice the 12 symmetries of A_4.

This phenomenon is called isomorphism.

They share the same elements. They are both groups. They are both subgroups of the symmetric group S_4.

A_4 =

1-cycles: (e)

3-cycles: (123), (124), (134), (132), (143), (142), (234), (243)

2x2- cycles: (12)(34), (13)(24), (14)(23)

Rot(T) =

Vertices: 1- (234), (243)

2- (134), (143)

3- (124), (142)

4- (123), (132)

Opposite Edges:

(12)(34)

(13)(24)

(14)(23)

The Identity: (e)

Isomorphism: A one-to-one correspondence between the elements of two sets such that the result of an operation on elements of one set corresponds to the result of the same operation in the other set.

	Rot (T)	A_4	
Rot(0,2π)	e	e	Identity

Rot($2\pi/3$, $4\pi/3$) (about each vertex)	Fixed: 1- Rotates face (2,3,4), (2,4,3) 2- Rotates face (1,3,4), (1,4,3) 3- Rotates face (1,2,4), (1,4,2) 4- Rotates face (1,2,3), (1,3,2)	(123), (124), (134), (132), (143), (142), (234), (243)	3- cycles
Rot(π) (about centers of opposite edges)	Rotates faces 1. (1,2)(3,4), 2. (13)(24), 3. (14)(23),	(12)(34), (13)(24), (14)(23)	2x2-cycles

The compositions of different rotations in Rot(T) correspond with the permutations in A_4

A rotation about different edges and vertices corresponds with permuting different cycles.

Rotation about vertex 1 by $2\pi/3$ composed with a rotation about opposite edges by π:

$(12)(34) \bullet (234) = (124)$

(124) is a rotation of $2\pi/3$ about vertex 3.

Composing the same 3-cycle with the same 2x2-cycles will yield that same end result.

There is closure and correspondence between the two groups.

There is a link between the composition of rotations and the composition of permutations that make Rot(T) and A_4 isomorphic.

$Rot_1 \bullet Rot_2 \Leftrightarrow A_1 \bullet A_2$

$A_1 \bullet A_2 \Leftrightarrow Rot_1 \bullet Rot_2$

Under isomorphism, A_4 has its own subgroup. The rotation of each face on the tetrahedron is a subgroup : S_3.

A_3 is a subgroup of A_4.

A_3 is the group of permutations thatrotate one face. Only one vertex is fixed.If the black vertex is fixed, we are left with red, orange and purple which can be denoted 1,2, and 3. This shows the permutations of order 3.

(123),(132). Those permutations exist in A_4 because A_3 is a subgroup of A_4.

In conclusion, the tetrahedron contains symmetries of a 2-D equilateral triangle (S_3), symmetries about the center of edges (2x2-cycles), and symmetries about different vertices (3-cycles) or in general (S_4).

4.3.7. The Cube, the Octahedron, and S_4

Since the Cube and Octahedron are duals of each other, we will only focus on the rotational symmetries of the Cube, {4,3}. The cube has a total of 8 vertices, 12 edges, and 6 faces. The total number of rotational symmetries in the Cube is 24. Here's why:

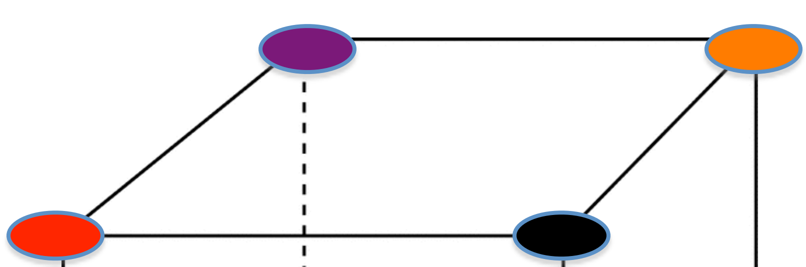
Possible symmetric rotations include rotating about opposite vertices, centers of opposite faces, and centers of opposite edges. This is because each of these rotations can divide evenly into 2π.

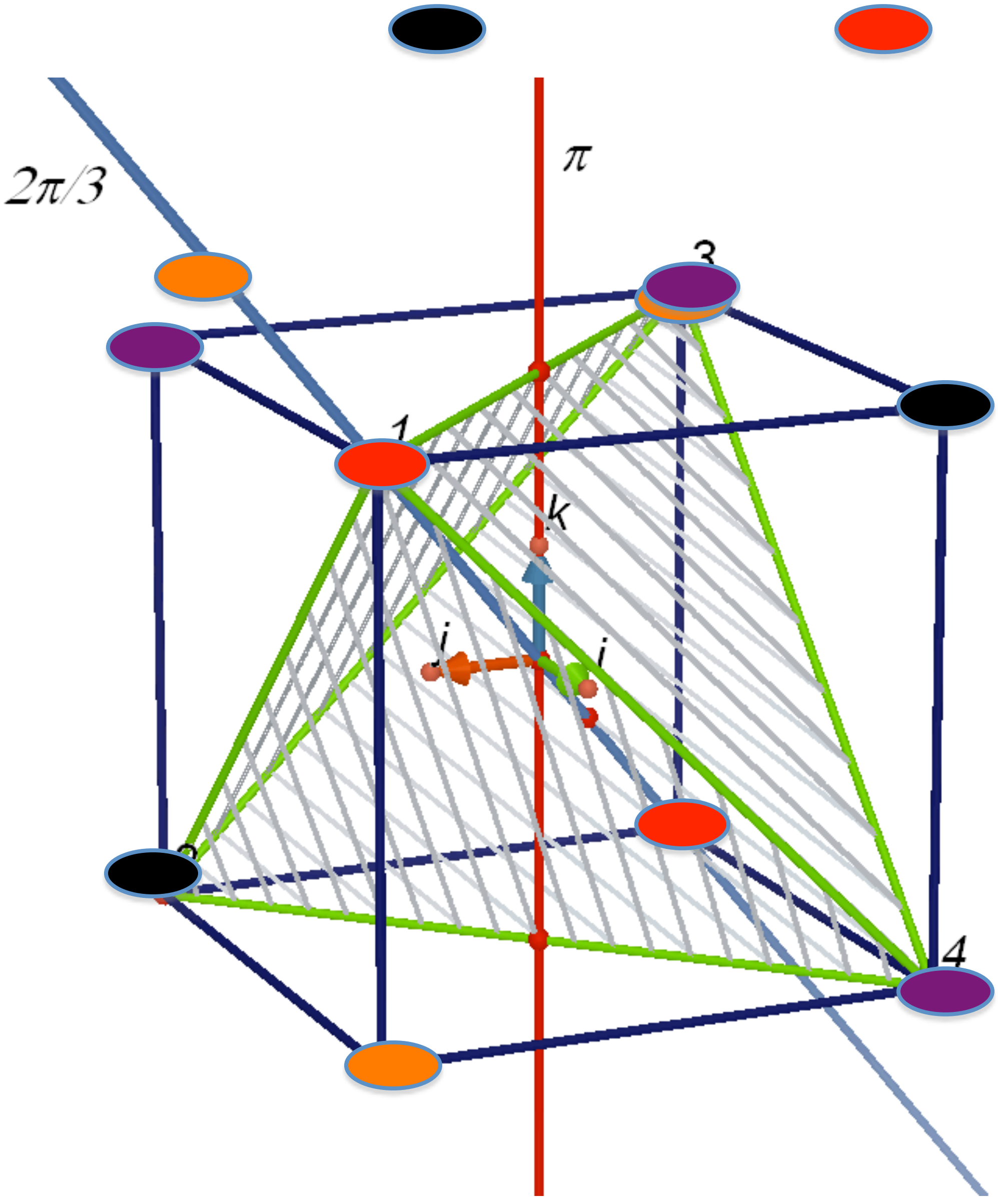
Let's first look at rotation by opposite vertices. Two π, or 360 degrees, is the degree of rotation that sends the points back to themselves. Throughout each symmetric rotation less than 2π, only point 1 will be sent to itself, while the rest are sent to new points. It takes three rotations about opposite vertices for this polyhedron to preserve the same points. Therefore, each symmetric rotation is a multiple of $2\pi/3$. In the Cube, there are four pairs of opposite vertices, and multiply that with the number of rotational symmetries less than 2π. This leaves us with 4 pairs of rotations*2 rotations ($2\pi/3$ and $4\pi/3$), which results in 8 total rotational symmetries about the opposite vertices. You apply this same method of thought for finding the total number of rotations by centers of opposite edges and by centers of opposite faces.

For the centers of opposite edges, two rotations are necessary for the polyhedron to be sent to itself. Therefore, the symmetric rotation is $2\pi/2$, which is reduced to π. Multiply the number of pairs of opposite edges (6) by total rotations under 2π (1), and the total rotational symmetries about the centers of opposite edges is 6.

For the centers of opposite faces, all vertices are being sent to new points, and it takes four rotations for the Cube to preserve the same shape with the same points before rotation, therefore each symmetric rotation is a multiple of $2\pi/4$, reduced to $\pi/2$. Multiply the number of pairs of faces (3) by total rotations under 2π (3), and the total rotational symmetries about the centers of the faces is 9. The identity counts as one rotational symmetry, so the total number of rotational symmetries for the cube is 24 (8+6+9+1).

S_4, a permutation group, is isomorphic to the cube. Let's see why:





Let’s label each colored vertex by a number.

Red=1 Orange=3

Purple=2 Black=4

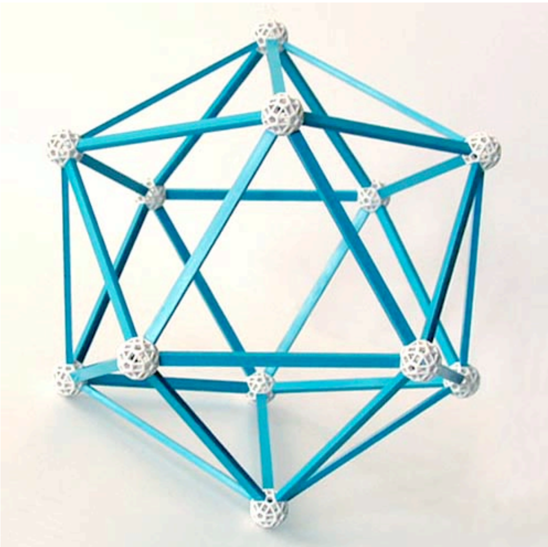
The total number of permutations of S_4 is 24. This set includes 4-cycles, 3-cycles, 2-cycles, 2*2-cycles, and 0-cycle. Four-cycles correspond to rotation of $\pi k/2$ about the centers of opposite faces, where k is an integer. Possible 4-cycles are (1234), (1243), (1324), (1342), (1423), and (1432). Two by two-cycles correspond to the rotation of πk about the centers of opposite faces. Possible 2*2-cycles are (12)(34), (13)(24), and (14)(23). Three-cycles correspond to the rotation of $2\pi k/3$ about opposite vertices. Possible 3-cycles are (123), (124), (132), (134), (142), (143), (234), and (243). Two-cycles correspond to the rotation of πk about opposite edges. Possible 2-cycles are (12), (13), (14), (23), (24), and (34). The 0-cycle corresponds to the rotation of e. In other words, if you simply label the vertices and rotate them by these angles, the permutations are all the possible orientations that would result. Because of the correspondence, Rot (C) and S_4 are isomorphic. Since this is the case, the subgroups of these two groups should also be isomorphic. Let's use the Tetrahedron, $A_{4,}$ and S_3 as the subgroups.

The Cube can have a Tetrahedron inscribed inside it, so the rotational symmetries inside the Tetrahedron are also included in the Cube. Rot (T) is isomorphic to A_4. Rot (C) includes rotations $2\pi k/3$ (about vertices and the centers of their opposite faces), π (about centers of opposite edges), and the identity. The Tetrahedron only has even cycles: 3-cycles, 2*2-cycles, and the 0-cycle, where the 3-cycle corresponds to rotation of $2\pi k/3$, the 2*2-cycle corresponds to rotation of π, and the 0-cycle corresponds to the rotation of e. Since A_4 includes only all of these even cycles, it is isomorphic to Rot (T). Another note to include is two tetrahedra can be inscribed inside the cube. Since the construction of the Tetrahedron only requires the opposite vertices of each face to be used, you can utilize the other unused set of opposite vertices to construct another one.

S3 is also a subgroup of S4. This is because it contains the 2-cycles, (12) (13) (23), the 3-cycles, (123) (132), and the0-cycle, which are all included in S4. S4 is isomorphic to the rotations (2-Cycles isomorphic toπ, 3-Cycles isomorphic to $2\pi k/3$, and 0-Cycle isomorphic to e). Note that point 4 is fixed in S3.

Rot (C)	<-------------------->	S_4
(6) Rotations by $\pi k/2$, about centers of opposite faces		(6) 4-cycles
(8) Rotations by $2\pi k/3$, about opposite vertices		(8) 3-Cycles
(6) Rotations by π about centers of opposite edges		(6) 2-Cycles
(3) Rotations by π about centers of opposite faces		(3) 2*2-Cycles

4.3.8 The Dodecahedron, the Icosahedron, and A_5

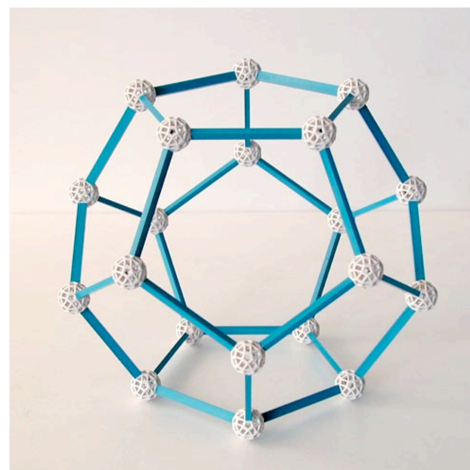


The dodecahedron (5, 3) is a polyhedra that consists of 20 vertices, 30 edges, and 12 faces. Its faces are pentagonal shaped and three of them meet at one vertex. Since the icosahedron is the dual of the dodecahedron, we will only focus on the dodecahedron. There are a total of 60 rotational symmetries of the dodecahedron. In a dodecahedron, you can rotate about centers of opposite faces (by $2\pi k/5$, where k is an integer less than 5), opposite vertices ($2\pi k/3$, where k is less than or equal to 2), opposite edges ($\pi/2$), and the identity (e).

The next question that arises is: how many symmetries are present in a dodecahedron? This is when we began using a counting principal which gave us this total. We first looked at the rotation upon the axes that form between opposite vertices. With the use of the counting principle we were able to see that there are a total of 20 symmetries that form from the pairs of opposite vertices. This 20 is logical because we know that there are 20 vertices. Since that is true then we know that there are 10 pairs of vertices which also means that there are 10 axes with two possible rotations each.(Rot($2\pi/3$) and Rot($4\pi/3$)) When you do the math two times ten you get 20 symmetries.

Another set of symmetries in the dodecahedron is that by the axes that go through the centers of opposite faces. Once again we use the counting principal to see that there are 24 of these symmetries. This can be seen as there are a total of 12 sides in a dodecahedron which also mean that there a total of 6 pairs of faces. This shows us that there are 6 more axes upon which you can rotate. Each of these axes has four possible rotations. These are the rotations by $2\pi/5$, $4\pi/5$, $6\pi/5$, and $8\pi/5$. When you do the multiplication four times six gives you a total of 24.

The final symmetries of the dodecahedron are those that pass through the centers of opposite edges. There are in total 15 of this type of symmetries. This is again seen because of the counting principle. There are a total of 30 edges and 15 pairs of edges which shows that there are 15 more axes upon which a dodecahedron can be rotated. There is only one rotation by π which can occur through each axis. When you do the multiplication 15 times one you get 15.



DODECAHEDRON

In total the dodecahedron has 60 symmetries: 20 symmetries by opposite vertices, 24 vertices through opposite faces, 15 symmetries through opposite edges, and finally the identity.

As previously seen, the rotational symmetries of certain poyhedra are isomorphic to groups of permutations. The reason we decided to look into these permutations in the first place was because closure would have been very difficult to show if we stuck to rotating the shape and seeing what happens. So instead we found that all the rotations of a polyhedra correspond to certain permutations. Since these permutations were easily composed we were able to prove closure. For the tetrahedron we saw that A_4, which is the alternating group of S_4, is isomorphic to the rotations of a tetrahedron. We also saw that S_4 is isomorphic to the rotations of a cube.

Now we had to look for a new group which is of order 60. As we had previously seen the order of S_5 had been 120 and we also knew that A_5 is half of that, or 60. Now that we had found the group of permutations that are of the same order we had to figure out if these two were indeed isomorphic.

We explicitly showed this by showing which rotations correspond with which permutations seen

	A_5	Rot(D)	count
Order 1	e the identity	Rot(0)	1
Order 2	2x2s ex: (12)(35)	Rot(π) about centers of opposite edges	15
Order 3	3-cycles ex: (135)	Rot(2 π/3) and Rot(4 π/3) about opposite pairs of vertices.	20
Order 5	5-cycles ex: (14325)	Rot(2 π/5), Rot(4 π/5), Rot (6 π/5), and Rot(8 π/5) about centers of opposite faces.	24

in the below chart: The next step we took was figuring out how to track the rotations of dodecahedron. For the tetrahedron we tracked the four vertices. For the square we tracked the four pairs of vertices since we figured out that the tetrahedron was a subgroup of the cube in terms of rotations. Now that we had somewhat a sense of what we were looking for we were able to see that the tetrahedron was also a subgroup of the dodecahedron. We saw this since the order of the tetrahedron was a factor of the order of the dodecahedron. We also saw that in terms of the permutations A_4 (isomorphic to the tetrahedron) is a subgroup of A_5. When you divide the order of the dodecahedron by the order of the tetrahedron you get five. This means that five tetrahedrons can be inscribed into a dodecahedron. We used this to our advantage by labeling each unique tetrahedron in the dodecahedron a different color. In the end we were basically permuting the tetrahedrons when we rotated the dodecahedron.

Now all that was left to do was to see that closure is present in $A_{5,}$ which it was. With that proof of closure we were done and we could see that the A_5 and the Rot(D) are isomorphic.

We then began to wonder that since A_4 was a subgroup of A_5 then was S_4 also a subgroup of $A_{5.}$ In order to answer our query we first experimented whether a cube could be inscribed into a dodecahedron, which it could, surprisingly. But, to our amazement S_4 was not actually a subgroup of A_5 as the orders of each could not be evenly divided into one another. This showed us that even if you can inscribe a shape into another that doesn't necessarily mean that it is a subgroup of the other.

4.3.9 The Finite Rotational Symmetries of the Sphere

In this project we looked at both the rotational symmetries and the reflective symmetries of regular polygons in the plane, as well as, the rotational symmetries in three-dimensional space. These symmetries are also similar to those of a circle (S^1) and the sphere (S^2).

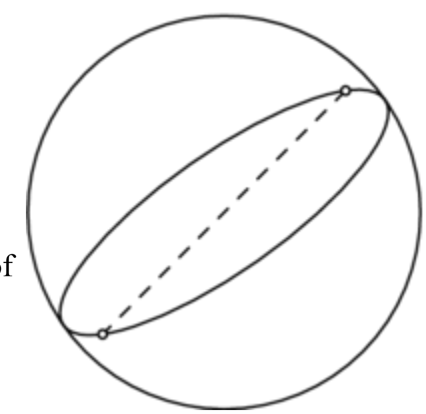
Theorem 1

Let $H \neq \{e\}$ be a non-trivial finite subgroup of $Sym(S^1)$. Then:

(i) H is cyclic of order 2, generated by a single reflection, or there exists a regular m-gon P_m such that

(ii) $H = Rot(P_m) = R_m$, cyclic of order m, consisting of the rotational symmetries of Pm, or

(iii) $H = Sym(P_m) = D_m$, the Dihedral group of order 2m, consisting of the full rotational and reflective symmetries of P_m.



With the above theorem we can see that All symmetries of S^1 can be viewed as rotations of S^2. This basically means that $Sym(S^1)$ is a subgroup of $Rot(S^2)$. This can be seen by choosing any great circle of the sphere (a circle whose radius is equal to that of the given sphere) you will be able to see that the reflections of the circle are also the same as a rotation by π through the diameter of that circle. This rotation by π will stay in the sphere.

In particular, all the finite subgroups of $\mathrm{Sym}(S^1)$ can actually be viewed as finite subgroups of $\mathrm{Rot}(S^2)$. This reminds us of another set of finite subgroups that we know of, which are the rotational symmetries of the Platonic solids - Rot(T), Rot(C) = Rot(O), and Rot(D) = Rot (I). The following theorem connects these two by saying that in addition to the finite subgroups of $\mathrm{Sym}(S^1)$ (the planar rotational symmetries), the only other finite subgroups of $\mathrm{Rot}(S^2)$ are precisely the rotation groups of the Platonic solids:

Theorem 2

Let $H \neq \{e\}$ be a non-trivial finite subgroup of $\mathrm{Rot}(S^2)$. Then:

(i) H is a planar rotational symmetry, i.e., a finite subgroup of $\mathrm{Sym}(S^1)$, or

(ii) H = Rot(P), where P is a Platonic solid.

These two theorems clearly unify Project I and Project II. The first explored the rotational symmetries of a circle, and the second dealt with the rotational symmetries in three dimensional space. However, rotational and reflective symmetries of a circle are actually rotational symmetries of a sphere, so Project I is actually just a special case of Project II. When you think about it the finite subgroups of the respective symmetries (Sym(S1) and Rot(S2))have only one difference between them, which is the rotation groups of the Platonic Solids, which were the main focus of Project II.

5. Conclusion

In conclusion, the objective of this project was to study geometric symmetry, as well as how closely it ties in with another branch, algebra. In Project I we explored the translation of algebra to geometry, while in Project II, we explored the translation of geometry to algebra.

Our first project began with the unit circle and simple planar symmetry of regular polygons. We recognized the two geometric symmetries in the 2-D plane, rotations and reflections. As we progressed, we investigated complex numbers as well as the complex plane and later we denoted the unit circle as a group. Upon our in-depth study of groups, we found that subgroups also existed within them. In order to be a subgroup of a group, four properties had to be satisfied. These were closure, associativity, identity, and inverse. Afterwards, we saw that a regular polygon representing a subgroup could be inscribed in another regular polygon of that group. Concluding Project I, we investigated dihedral groups and finite symmetries of the circle.

After discovering 2-D rotational symmetries of polygons, that led us to study 3-D rotational symmetries of polyhedra, Project II. To help us study 3-D rotations, we studied Platonic solids, which consisted of five figures and were fairly distinct. Upon studying 3-D rotational symmetries, we found another way to express the rotations of a Platonic solid, algebraically through group theory, which we further concluded was equivalent to its geometric representation. To find the algebraic counter partners, we studied permutations and returned to groups and subgroups, once again showing the relationship of both geometry and algebra.

In the words of Henri Poincaré:

"Mathematics is the art of giving the same name to different things; poetry is the art of giving different names to the same thing."

References:

Artin, Michael. *Algebra*. Boston, MA: Pearson Education, 2011. Print.

Coxeter, H. S. M. *Introduction to Geometry, Second Edition*. New York: Wiley, 1989. Print.

Coxeter, H. S. M., and Samuel L. Greitzer. *Geometry Revisited*. New York: Mathematical Assoc. of America, 2008. Print.

Coxeter, H. S. M. *Regular Polytopes*. New York: Dover Publications, 1973. Print.

Dummit, David S., and Richard M. Foote. *Abstract Algebra*. Hoboken, NJ: Wiley, 2004. Print.

Flanigan, Francis J. *Complex Variables: Harmonic and Analytic Functions*. New York: Dover Publications, 1983. Print.

Hillman, Abraham P., Gerald L. Alexanderson, and Abraham P. Hillman. *Abstract Algebra: A First Undergraduate Course*. Prospect Heights, IL: Waveland, 1999. Print.

Pinter, Charles C. *A Book of Abstract Algebra*. Mineola, N.Y: Dover Publications, 2010. Print.

Pedoe, Daniel. *Geometry, a Comprehensive Course*. New York: Dover Publications, 1988. Print.

Silverman, Richard A. *Introductory Complex Analysis.* N.p.: Dover, 1984. Print.

Stewart, Ian. *Concepts of Modern Mathematics*. New York: Dover, 1995. Print.

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