The Cooper Union
Department of Electrical Engineering
ECE114 Digital Signal Processing
Lecture Notes: Sinc Functions and Sampling Theory
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Sinc Functions

A rectangular pulse in time/frequency corresponds to a *sinc* function in frequency/time. Two sinc functions arise: the "ordinary" sinc, essentially $\sin\theta/\theta$, which extends from $-\infty$ to ∞ and has equally spaced zero crossings, and the *Dirichlet sinc,* which is periodic and also has equally spaced zero crossings.

Here, we take:

$$\operatorname{sinc}\theta = \frac{sin\theta}{\theta}$$

Note that sometimes the sinc function is defined as $\operatorname{sinc}(x) = \frac{sin\pi x}{\pi x}$, as in MATLAB. The Dirichlet sinc of order N is defined as:

$$D_N(\omega) = \frac{sin(N\omega/2)}{N sin(\omega/2)}$$

Continuous-Time Rectangular Pulse

Let $x(t) = 1$, $0 \le t \le T$, and 0 otherwise. Then:

$$X(f) = e^{-j\pi fT} T \frac{\sin \pi fT}{\pi fT}$$

Notice the linear-phase factor arises because $x(t)$ is symmetric about time $T/2$. That is, $x(t) = x_0(t - T/2)$ where $X_0(f)$ is zero-phase. Also observe that:

$$X(mf_0) = T\delta(m)$$

where here $\delta(\cdot)$ is the discrete-time impulse.

$$f_0 = 1/T$$

We verify the result:

$$\begin{aligned} X(f) &= \int_0^T e^{-j2\pi ft} dt \\ &= \frac{1}{-j2\pi f} e^{-j2\pi ft}\Big|_0^T \\ &= \frac{1 - e^{-j2\pi fT}}{j2\pi f} \\ &= e^{-j\pi fT} T \frac{e^{j\pi fT} - e^{-j\pi fT}}{2j(\pi fT)} \\ &= e^{-j\pi fT} T \frac{\sin \pi fT}{\pi fT} \end{aligned}$$

Ideal Analog Lowpass Filter

Let $H(f) = 1$, $|f| \leq f_c$, $H(f) = 0$ otherwise. Then:

$$h(t) = 2f_c \frac{sin\pi t/T}{\pi t/T}$$

where $T = 1/(2f_c)$. Observe that:

$$h(nT) = \delta(n)$$

where $\delta(\cdot)$ is the discrete-time impulse.

We verify the result:

$$\begin{aligned} h(t) &= \int_{-f_c}^{f_c} e^{j2\pi ft} df \\ &= \frac{1}{j2\pi t}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right] \\ &= 2f_c \frac{sin(2\pi f_c t)}{2\pi f_c t} \end{aligned}$$

Ideal Digital Lowpass Filter

Let $H(\omega) = 1$, $|\omega| \leq \omega_c$, $H(\omega) = 0$ otherwise. Then:

$$h(n) = \frac{\omega_c}{\pi}\frac{sin(n\omega_c)}{n\omega_c}$$

Observe that $h(n)$ is a sampled sinc function, but does not necessarily have exact zero-crossings (if those zero-crossings would not occur at integer time points).

We verify the result:

$$\begin{aligned} h(n) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \\ &= \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right] \\ &= \frac{\omega_c}{\pi}\frac{sin(n\omega_c)}{n\omega_c} \end{aligned}$$

Discrete-Time Rectangular Pulse

Let $h(n) = 1$, $0 \leq n \leq N-1$, and $h(n) = 0$ otherwise. This is a discrete-time rectangular pulse. We would expect its DTFT $H(\omega)$ to be a sinc function. However, $H(\omega)$ must be *periodic.* Therefore, it is a *Dirichlet sinc.* Specifically:

$$H(\omega) = e^{-j\left(\frac{N-1}{2}\right)\omega} N \frac{sin(N\omega/2)}{N sin(\omega/2)} = e^{-j\left(\frac{N-1}{2}\right)\omega} N D_N(\omega)$$

Observe that:

$$H(k\omega_0) = N\delta(k \bmod N) = \begin{cases} N & k = 0, \pm N, \pm 2N, \cdots \\ 0 & \text{otherwise} \end{cases}$$

where $\delta(\cdot)$ is the discrete impulse and $\omega_0 = 2\pi/N$. In other words, $H(\omega)$ is zero at all the DFT bin frequencies, other than DC.

To see this:

$$\begin{aligned} H(\omega) &= \sum_{0}^{N-1} e^{-j\omega n} \\ &= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} \\ &= \frac{e^{-j\omega N/2}}{e^{-j\omega/2}} \frac{e^{j\omega N/2} - e^{-j\omega N/2}}{e^{j\omega/2} - e^{-j\omega/2}} \\ &= e^{-j\left(\frac{N-1}{2}\right)\omega} \frac{sin(N\omega/2)}{sin(\omega/2)} \\ &= e^{-j\left(\frac{N-1}{2}\right)\omega} N D_N(\omega) \end{aligned}$$

Other cases

Suppose we have a *periodic continuous-time* pulse train with period T, with $x(t) = 1$, $0 \leq t \leq \tau_0$, 0 for $\tau_0 < t \leq T$. Then the line spectrum are samples of a sinc. Which sinc?

Suppose we have a set of DFT coefficients given by $X(k) = 1$, $0 \leq k \leq M$, and $X(k) = 0$, $M + 1 \leq k \leq N - 1$. Then $x(n)$ are samples of a sinc. Which sinc?

Impulse Train

Consider a continuous-time periodic impulse train:

$$\sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Theorem 1 *The Fourier transform of a periodic impulse train is a periodic impulse train. Specifically:*

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \longleftrightarrow f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$$

where $f_s = 1/T$.

Proof. The Fourier transform of the impulse train is:

$$\mathcal{F} \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} e^{-j2\pi f nT}$$

Now, consider a discrete-time signal $x(n) = 1$ for all n. Then its DTFT is $X(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega n}$. We know this to be $2\pi\delta(\omega)$ via the IDTFT formula:

$$1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} [2\pi\delta(\omega)] e^{j\omega n} d\omega$$

However, let us be more precise about this. On the one hand:

$$\sum_{-\infty}^{\infty} e^{-j\omega n} = 2\pi\delta\left(\omega\right), \ -\pi \leq \omega \leq \pi$$

On the other hand, we know this function is periodic with period 2π. Thus, the more precise result is:

$$\sum_{-\infty}^{\infty} e^{-j\omega n} = 2\pi \sum_{m=-\infty}^{\infty} \delta\left(\omega - 2\pi m\right), \ -\infty \leq \omega \leq \infty$$

Now, let us consider $\delta\left(a\xi\right)$ in terms of $\delta\left(\xi\right)$, for $a \neq 0$. Observe:

$$\int_{\xi=-\infty}^{\infty} \delta\left(a\xi\right) f\left(\xi\right) d\xi = \frac{1}{|a|} \int_{\zeta=-\infty}^{\infty} \delta\left(\zeta\right) f\left(\zeta/a\right) d\zeta = \frac{1}{|a|} f\left(0\right)$$

where we make the substitution $\zeta = a\xi$. Thus:

$$\delta\left(a\xi\right) = \frac{1}{|a|}\delta\left(\xi\right)$$

Therefore:

$$\begin{aligned} \sum_{n=-\infty}^{\infty} e^{-j2\pi f nT} &= 2\pi \sum_{m=-\infty}^{\infty} \delta\left(2\pi f T - 2\pi m\right) \\ &= 2\pi \frac{1}{2\pi T} \sum_{m=-\infty}^{\infty} \delta\left(f - 2\pi m/2\pi T\right) \\ &= f_s \sum_{m=-\infty}^{\infty} \delta\left(f - m f_s\right) \end{aligned}$$

with $f_s = 1/T$. ■

Sinc Interpolation Formula

The basic sampling theorem is that if:

$$x_a\left(t\right) \longleftrightarrow X_a\left(f\right)$$

and $x\left(n\right) = x_a\left(nT\right)$, and:

$$x\left(n\right) \longleftrightarrow X\left(\omega\right)$$

then:

$$X\left(\omega\right) = f_s \sum_{m=-\infty}^{\infty} X_a\left(f - m f_s\right)\Big|_{\omega = 2\pi f/f_s}$$

where $f_s = 1/T$. In terms of analog radian frequency $\Omega = 2\pi f$, with $\Omega_s = 2\pi f_s = 2\pi/T$:

$$X\left(\omega\right) = f_s \sum_{m=-\infty}^{\infty} X_a\left(\Omega - m\Omega_s\right)\Big|_{\omega = \Omega T}$$

One question we may have is what is the *analog* signal whose spectrum is periodic. The answer is an impulse train, scaled by the samples of $x_a\left(t\right)$. That is:

Theorem 2 *The inverse CTFT of* $f_s \sum_{-\infty}^{\infty} X_a(f - mf_s)$, *i.e., the* analog *signal whose spectrum is the periodized version of* $X(f)$, *is:*

$$\sum_{n=-\infty}^{\infty} x(n)\,\delta(t - nT)$$

Proof. Consider $f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s)$. Its ICTFT is $\sum_{n=-\infty}^{\infty} \delta(t - nT)$. Multiplying the impulse train in the time domain by $x_a(t)$ yields:

$$x_a(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x_a(nT)\,\delta(t - nT) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t - nT)$$

But this corresponds to convolution in the frequency domain:

$$X_a(f) * f_s \sum_{m=-\infty}^{\infty} \delta(f - mf_s) = f_s \sum_{m=-\infty}^{\infty} X(f - mf_s)$$

■

Assuming $X_a(f)$ is bandlimited to the range $|f| \leq f_s/2$, we have:

$$X_a(f) = \sum_{-\infty}^{\infty} X_a(f - mf_s), \; -f_s/2 \leq f \leq f_s/2$$

Now suppose we apply the ideal brickwall filter $H(f)$:

$$H(f) = \begin{cases} 1 & |f| \leq f_s/2 \\ 0 & \text{otherwise} \end{cases}$$

Then:

$$X_a(f) = H(f) \cdot \sum_{-\infty}^{\infty} X_a(f - mf_s), \; -\infty < f < \infty$$

Taking the inverse Fourier transform yields:

Theorem 3 (Sinc Interpolation Formula) *If* $x_a(t)$ *is bandlimited to* $|f| \leq f_s/2$, *then it can be perfectly reconstructed from its samples* $x(n) = x_a(nT)$ *via:*

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)\,\phi(t - nT)$$

where:

$$\phi(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$