The Cooper Union
Department of Electrical Engineering
ECE135 Engineering Electromagnetics
Vector Analysis
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Vector Calculus Identities

$$
\begin{aligned}
\vec{A} \times \vec{B} &= -\vec{B} \times \vec{A} \\
\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) &= \vec{B} \cdot \left(\vec{C} \times \vec{A}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) \\
\vec{A} \times \left(\vec{B} \times \vec{C}\right) &= \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla\left(fg\right) &= f\nabla g + g\nabla f \\
\nabla \cdot \left(f\vec{G}\right) &= f\nabla \cdot \vec{G} + \vec{G} \cdot \nabla f \\
\nabla \times \left(f\vec{G}\right) &= f\nabla \times \vec{G} + \nabla f \times \vec{G} \\
\nabla \cdot \left(\vec{A} \times \vec{B}\right) &= \vec{B} \cdot \left(\nabla \times \vec{A}\right) - \vec{A} \cdot \left(\nabla \times \vec{B}\right)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \nabla f &= 0 \\
\nabla \cdot \left(\nabla \times \vec{G}\right) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla^2 f &= \nabla \cdot \nabla f \\
\nabla^2 \vec{G} &= \nabla\left(\nabla \cdot \vec{G}\right) - \nabla \times \left(\nabla \times \vec{G}\right)
\end{aligned}
$$

$$
\begin{aligned}
\int_V \nabla \cdot \vec{G} dv &= \oint_{\partial V} \vec{G} \cdot \vec{n} dS \\
\int_V \nabla \times \vec{G} dv &= \oint_{\partial V} \vec{n} \times \vec{G} dS \\
\int_S \left(\nabla \times \vec{G}\right) \cdot \vec{n} dS &= \oint_{\partial S} \vec{G} \cdot \vec{T} ds
\end{aligned}
$$

Rectangular Coordinates: (x, y, z)

$$\vec{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$$

$$\begin{aligned}
\hat{a}_x \times \hat{a}_y &= \hat{a}_z \\
\hat{a}_y \times \hat{a}_z &= \hat{a}_x \\
\hat{a}_z \times \hat{a}_x &= \hat{a}_y
\end{aligned}$$

Cylindrical Coordinates: (r, ϕ, z)

$$\vec{r} = r\hat{a}_r + z\hat{a}_z$$

$$\begin{aligned}
\hat{a}_r \times \hat{a}_\phi &= \hat{a}_z \\
\hat{a}_\phi \times \hat{a}_z &= \hat{a}_r \\
\hat{a}_z \times \hat{a}_r &= \hat{a}_\phi
\end{aligned}$$

$$\begin{aligned}
\hat{a}_r &= \cos\phi\hat{a}_x + \sin\phi\hat{a}_y \\
\hat{a}_\phi &= -\sin\phi\hat{a}_x + \cos\phi\hat{a}_y \\
\hat{a}_z &= \hat{a}_z
\end{aligned}$$

$$\begin{aligned}
x &= r\cos\phi \\
y &= r\sin\phi \\
z &= z
\end{aligned}$$

Spherical Coordinates: (r, ϕ, θ)

$$\vec{r} = r\hat{a}_r$$

$$\begin{aligned}
\hat{a}_r \times \hat{a}_\theta &= \hat{a}_\phi \\
\hat{a}_\theta \times \hat{a}_\phi &= \hat{a}_r \\
\hat{a}_\phi \times \hat{a}_r &= \hat{a}_\theta
\end{aligned}$$

$$\begin{aligned}
\hat{a}_r &= \sin\theta\left(\cos\phi\hat{a}_x + \sin\phi\hat{a}_y\right) + \cos\theta\hat{a}_z \\
\hat{a}_\phi &= \sin\theta\left(-\sin\phi\hat{a}_x + \cos\phi\hat{a}_y\right) \\
\hat{a}_\theta &= \cos\theta\left(\cos\phi\hat{a}_x + \sin\phi\hat{a}_y\right) - \sin\theta\hat{a}_z
\end{aligned}$$

$$\begin{aligned}
x &= r\cos\phi\sin\theta \\
y &= r\sin\phi\sin\theta \\
z &= r\cos\theta
\end{aligned}$$

Differential Forms and Operators

Rectangular Coordinates: (x, y, z)

$$\begin{aligned}
\nabla f &= \hat{a}_x \frac{\partial f}{\partial x} + \hat{a}_y \frac{\partial f}{\partial y} + \hat{a}_z \frac{\partial f}{\partial z} \\
\nabla \cdot \vec{G} &= \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} \\
\nabla \times \vec{G} &= \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ G_x & G_y & G_z \end{vmatrix} \\
&= \left(\frac{\partial G_z}{\partial y} - \frac{\partial G_y}{\partial z} \right) \hat{a}_x + \left(\frac{\partial G_x}{\partial z} - \frac{\partial G_z}{\partial x} \right) \hat{a}_y + \left(\frac{\partial G_y}{\partial x} - \frac{\partial G_x}{\partial y} \right) \hat{a}_z \\
\nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2 \vec{G} &= \nabla^2 G_x \hat{a}_x + \nabla^2 G_y \hat{a}_y + \nabla^2 G_z \hat{a}_z
\end{aligned}$$

Differential volume : $d\text{vol} = dxdydz$

Cylindrical Coordinates: (r, ϕ, z)

$$\begin{aligned}
\nabla f &= \hat{a}_r \frac{\partial f}{\partial r} + \hat{a}_\phi \frac{1}{r} \frac{\partial f}{\partial \phi} + \hat{a}_z \frac{\partial f}{\partial z} \\
\nabla \cdot \vec{G} &= \frac{1}{r} \frac{\partial}{\partial r} (r G_r) + \frac{1}{r} \frac{\partial G_\phi}{\partial \phi} + \frac{\partial G_z}{\partial z} \\
\nabla \times \vec{G} &= \frac{1}{r} \begin{vmatrix} \hat{a}_r & r\hat{a}_\phi & \hat{a}_z \\ \partial/\partial r & \partial/\partial \phi & \partial/\partial z \\ G_r & rG_\phi & G_z \end{vmatrix} \\
&= \left(\frac{1}{r} \frac{\partial G_z}{\partial \phi} - \frac{\partial G_\phi}{\partial z} \right) \hat{a}_r + \left(\frac{\partial G_r}{\partial z} - \frac{\partial G_z}{\partial r} \right) \hat{a}_\phi + \frac{1}{r} \left(\frac{\partial}{\partial r} (r G_\phi) - \frac{\partial G_r}{\partial \phi} \right) \hat{a}_z \\
\nabla^2 f &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}
\end{aligned}$$

Differential volume:

$$d\text{vol} = r dr d\phi dz$$

Differential surface area (constant r):

$$dS = r d\phi dz$$

Spherical Coordinates: (r, θ, ϕ)

$$\nabla f = \hat{a}_r \frac{\partial f}{\partial r} + \hat{a}_\theta \frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{a}_\phi \frac{1}{r \sin\theta}\frac{\partial f}{\partial \phi}$$

$$\nabla \cdot \vec{G} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 G_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(G_\theta \sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial G_\phi}{\partial \phi}$$

$$\begin{aligned}\nabla \times \vec{G} &= \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{a}_r & r\hat{a}_\theta & r\sin\theta \hat{a}_\phi \\ \partial/\partial r & \partial/\partial \theta & \partial/\partial \phi \\ G_r & rG_\theta & r\sin\theta G_\phi \end{vmatrix} \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}\left(G_\phi \sin\theta\right) - \frac{\partial G_\theta}{\partial \phi}\right)\hat{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial G_r}{\partial \phi} - \frac{\partial}{\partial r}\left(rG_\phi\right)\right)\hat{a}_\theta \\ &\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(rG_\theta\right) - \frac{\partial G_r}{\partial \theta}\right)\hat{a}_\phi\end{aligned}$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

Differential volume:

$$d\text{vol} = r^2 \sin\theta dr d\theta d\phi$$

Differential surface area (constant r):

$$dS = r^2 \sin\theta d\theta d\phi$$

Free-Space Constants

Speed of light: $c \approx 3 \times 10^8 \ \ m/\sec$

Permittivity: $\epsilon_0 \equiv \frac{1}{\mu_0 c^2} \approx \frac{1}{36\pi} \times 10^{-9} \approx 8.854 \times 10^{-12} \ \ F/m$

Permeability: $\mu_0 \equiv 4\pi \times 10^{-7} \ \ H/m$

Impedance: $\eta_0 \equiv \sqrt{\mu_0/\epsilon_0} \approx 120\pi \approx 377 \ \ \Omega$

Additional Remarks

r and $\hat{a}_r$ for cylindrical coordinates are **not** the same as r and $\hat{a}_r$ for spherical coordinates. Some texts use different symbols (e.g., r for cylindrical case and R for spherical case); we will just be clear to indicate the coordinate system in use.

$\hat{a}_\phi$ *does* denote the same vector in either cylindrical or spherical coordinates, and the parameter ϕ is the same in both cases as well.

It is important to recognize that the vectors $\hat{a}_r, \hat{a}_\phi$ in cylindrical coordinates and $\hat{a}_r, \hat{a}_\phi, \hat{a}_\theta$ in spherical coordinates are not constant, but are *spatially varying.* This causes formulas involving ∇ to become complicated. Specifically: $\hat{a}_\phi = \hat{a}_\phi(\phi)$ depends on ϕ only. In cylindrical coordinates, $\hat{a}_r = \hat{a}_r(\phi)$ depends on ϕ, not r or z. In spherical coordinates, $\hat{a}_r = \hat{a}_r(\theta, \phi)$ and $\hat{a}_\theta = \hat{a}_\theta(\theta, \phi)$ depend on both angle parameters, but not r.

For a vector field $\vec{G}(\vec{r})$, $\nabla^2 \vec{G} = \left(\nabla^2 G_x\right) \hat{a}_x + \left(\nabla^2 G_y\right) \hat{a}_y + \left(\nabla^2 G_z\right) \hat{a}_z$ in **rectangular coordinates!!** In the other coordinate systems, since the unit vectors are spatially varying, similar decompositions are not correct; for example, in spherical coordinates:

$$\nabla^2 \vec{G} \neq \left(\nabla^2 G_r\right) \hat{a}_r + \left(\nabla^2 G_\phi\right) \hat{a}_\phi + \left(\nabla^2 G_\theta\right) \hat{a}_\theta$$

The correct formulas are quite complicated, and are not included in these notes. The coordinate-free definition of $\nabla^2 \vec{G}$ is:

$$\nabla^2 \vec{G} = \nabla \left(\nabla \cdot \vec{G}\right) - \nabla \times \left(\nabla \times \vec{G}\right)$$

θ is called the *polar* angle and ϕ the *azimuth.* $\theta = 0$ at the "north pole" and $\theta = \pi$ at the "south pole." In some situations, it is more convenient to work with $\theta' = \pi/2 - \theta$, called the *elevation* angle; $\theta' = 0$ in the xy-plane, $\theta' > 0$ in the upper hemispher, and $\theta' < 0$ in the lower hemisphere.

In general, a vector field is specified uniquely up to an additive constant by its divergence $\nabla \cdot \vec{F} = g(\vec{r})$ and curl $\nabla \times \vec{F} = \vec{G}(\vec{r})$; g and $\vec{G}$ can be specified arbitrarily (i.e., independently of each other). Similarly, a scalar field is uniquely determined, up to an additive constant, by its gradient $\nabla f = \vec{G}(\vec{r})$.