The Cooper Union
Department of Electrical Engineering
ECE114 Digital Signal Processing
Lecture Notes: Complex Analysis
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Complex Numbers

$z = x + jy = re^{j\theta}$ where $e^{j\theta} = \cos\theta + j\sin\theta$. Note $e^{j2n\pi} = 1$ iff n is an integer.

conjugate: $\bar{z} = z^* = x - jy = re^{-j\theta}$

$|z|^2 = zz^*$, $\theta =$argument of $z = \arg(z)$

The *unit circle* is $|z| = 1$.

$\log z = \ln|z| + j\arg(z) = \ln r + j\theta$

Actually, $\log z$ is multiple valued: $\log z = \ln r + j\theta + j2n\pi$, $n=$ integer.

Note the natural logarithm is written ln for real parameters and log for complex. In MATLAB, *log, abs* and *angle* return the (complex or real) natural log, magnitude and argument, respectively.

If z, w are complex: $z^w = e^{w\log z}$. Thus in general z^w is multiple valued, unless $w = k =$ integer: z^k is single valued.

Example 1 *$j^j = e^{j(j\pi/2+j2n\pi)} = e^{-\pi/2}e^{-2n\pi}$, so j^j has ∞ many values, all positive real, with values arbitrarily small ($n \to +\infty$) and arbitrarily large ($n \to -\infty$).*

$z^{1/N} = e^{(\ln r+j\theta+j2n\pi)/N} = r^{1/N}e^{j\theta/N}e^{j2\pi n/N}$, which yields N distinct values for $0 \le n \le N-1$. The N values $e^{j2\pi n/N}$ are the distinct N^{th} roots of unity (α is an N^{th} root of unity if $\alpha^N = 1$), and are equally spaced around the unit circle. The *twiddle factor* is $W_N = e^{-j2\pi/N}$ is defined in the context of DFT/FFT. It is a *primitive* N^{th} root of unity, so-called because its powers W_N^k, $0 \le k \le N-1$, are all the distinct N^{th} roots of unity.

Example 2 $\left(8e^{j\pi/4}\right)^{1/3} = 2e^{j\pi/12} \times 1, \times e^{j2\pi/3}, \times e^{-j2\pi/3}$

Analytic Functions and Singularities

A complex-valued function of a complex variable $f(z)$ can be expressed as:

$$f(z) = u(x,y) + jv(x,y)$$

where u, v are each real-valued functions of two real variables, x, y. The derivative f' of f is defined as:

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If f' exists at z_o, then f is *analytic* at z_o. If f' exists at all points within some region, then f is said to be analytic in the region. If f is analytic for all z, f is called *entire.* For example, polynomials and e^z are entire.

The existence of the complex derivative f' is a much stronger condition than in the case of real functions. Specifically, as $\Delta z \to 0$ in any way, the limit must always yield the same answer.

Example 3 *z^*, $|z|$, and most general expressions involving conjugate or magnitude are not analytic for any z. Let $f(z) = z^* = x - jy$. First, if $\Delta z = \Delta x$ purely real:*

$$\frac{f(z+\Delta x) - f(z)}{\Delta x} = \Delta x / \Delta x \to +1$$

But if $\Delta z = j\Delta y$ is purely imaginary then:

$$\frac{f(z+j\Delta y) - f(z)}{j\Delta y} = -j\Delta y / j\Delta y \to -1$$

Polynomials, ratios of polynomials, exponential and trigonometric functions are analytic except at points where the denominator goes to zero. For example, $\tan z = \sin z / \cos z$ is not analytic at $z = \pi/2 + n\pi$, $n=$ integer.

If f is analytic in a region, it may be possible to extend f beyond this region so it stays analytic. This is called *analytic continuation.* The analytic continuation is uniquely determined by the values on the boundary.

Example 4 *$f(z) = \sum_{n=0}^{\infty} z^n$ is analytic for $|z| < 1$, but the series does not converge[1] for $|z| \geq 1$. However, we can use the formula $1/(1-z)$ to perform analytic continuation to all $z \neq 1$. That is, $\hat{f}(z) = 1/(1-z)$ has the property that $\hat{f} = f$, $|z| < 1$, and $\hat{f}$ is defined and analytic for $|z| \geq 1$, $z \neq 1$ as well as for $|z| < 1$. For example, $f(2) = \sum_{n=0}^{\infty} 2^n$ is nonsense but the analytic continuation $\hat{f}(2) = \frac{1}{1-2} = -1$ is perfectly valid.*

$f(z) = u(x,y) + jv(x,y)$ is analytic at z iff u, v satisfy the *Cauchy-Riemann (C-R) equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 5 ***singularity:*** *If f is analytic in a region except at an isolated point z_o, then f has a* singularity *at $z = z_o$.*

Theorem 6 ***types of singularities:*** *There are only three types of singularities, which can be classified according to the behavior of* $\lim_{z \to z_o} |f(z)|$*:*

1. removable singularity: $\lim_{z \to z_o} f(z)$ *exists. If $f(z_o)$ is set equal to this limit, f becomes analytic there. For example, $f(z) = \sin z / z$ is analytic at $z = 0$ if we define $f(0) = 1$, so $z = 0$ is a removable singularity.*

1. *pole:* $\lim_{z \to \infty} |f(z)| = \infty$. Then f must blow up with polynomial growth: $f(z) = g(z) / (z - z_o)^m$ where g is analytic at z_o, $g(z_o)$ being nonzero finite. Equivalently:

$$\lim_{z \to z_o} (z - z_o)^k f(z) = \begin{cases} 0, & k > m \\ \text{finite, not } 0, & k = m \\ \text{blows up}, & k < m \end{cases}$$

The *multiplicity* of the pole is m.

[1] Here, "convergence" of a series or integral ALWAYS refers to absolute convergence.

2. *essential singularity:* $\lim_{z\to z_o}|f(z)|$ does not exist: it is neither a finite constant nor ∞. For example, $e^{1/z}\to 0$ if $z=x$, $x<0$, $x\to 0$, and $e^{1/z}\to\infty$ if $z=x$, $x>0$, $x\to 0$, but $\left|e^{1/z}\right|=1$ if $z=jy$. Thus, $\lim_{z\to 0}|f(z)|$ does not exist.

Theorem 7 ***Behavior at an essential singularity (Picard's theorem):*** *If z_o is an essential singularity, then in any arbitrarily small neighborhood of z_o, f takes on all complex values infinitely often, except for exactly one value.*

Example 8 *$f(z)=e^{1/z}$ has an essential singularity at $z=0$. For all $\epsilon>0$ and all $w\neq 0$, we can find $|z|<\epsilon$ such that $f(z)=w$. Setting $e^{1/z}=w$, $w\neq 0$, we get:*

$$z=1/\log w=\frac{1}{\ln|w|+j\theta+j2n\pi}$$

By picking $|n|$ large enough we can find a z with arbitrarily small magnitude. Note $e^{1/z}\neq 0$ for any z.

To repeat, these are the *only* possibilities! If f is analytic in a region except for poles, it is called homomorphic. If $f(z)=(z-z_o)^m g(z)$ where g is analytic at z_o and $g(z_o)\neq 0$, then f has a *zero* at z_o of multiplicity m. That is:

$$\lim_{z\to z_o}\frac{f(z)}{(z-z_o)^k}=\begin{cases} 0, & k<m\\ \text{finite, not } 0, & k=m\\ \text{blows up}, & k>m\end{cases}$$

Similar to the concept of multiplicity of poles is the multiplicity of zeros.

Problem 9 ***Theorem 10 zero of an analytic function:*** *If f is analytic at z_0 and $f(z_0)=0$, then either $f(z)$ is* ***identically*** *zero everywhere, or $f(z)=(z-z_0)^m g(z)$ where g is analytic at z_0 and $g(z_0)\neq 0$; m is called the* multiplicity *of the zero.*

In complex analysis, ∞ is considered as a single point. The expression $z\to\infty$ means that $|z|\to\infty$, the behavior of the $\arg z$ can be arbitrary. We say f is analytic at ∞, has poles or zeros at ∞, etc., according to its behavior as $|z|\to\infty$. Specifically:

Theorem 11 ***behavior at ∞:*** *We say $f(z)$ is analytic at ∞ if $f(z)$ is analytic for all finite $|z|>R$, for some finite radius R, and $\lim_{z\to\infty}f(z)$ exists (in this case we denote the limit as $f(\infty)$). The function f has a zero at ∞ with multiplicity m if it is analytic there and $f(z)=g(z)/z^m$ where g is analytic at ∞ and $g(\infty)\neq 0$; if $f(\infty)=0$, then either there is finite multiplicity or f is identically zero. The function f has a pole at ∞ if it is analytic for all finite $|z|>R$, for some finite radius R, and $\lim_{z\to\infty}|f(z)|=\infty$; in this case, $f(z)=z^m g(z)$ where g is analytic at infinity and $g(\infty)\neq 0$, and m is called the multiplicity. If f is analytic for all $|z|>R$ for some finite R but $\lim_{z\to\infty}|f(z)|$ does not exist and is not ∞, then f has an essential singularity at ∞, and it satisfies the behavior described in Picard's theorem (i.e., it takes on all finite complex values except exactly one in every region $|z|>R$).*

Example 12 *Consider:*

$$f_1(z) = e^z,\ f_2(z) = \frac{z^3+1}{2z+3},\ f_3(z) = \frac{z^3+1}{2z^3+3},\ f_4(z) = \frac{z^3+1}{2z^5+3}$$

Then f_1 has an essential singularity at ∞ (f_1 is still called entire); f_2 has a double pole at ∞ (it blows up like z^2; $f_2(z)/z^2 \to 1/2$ as $z \to \infty$); f_3 and f_4 are both analytic at ∞, with f_4 having a double zero there (it dies out like $1/z^2$; $z^2 f_4(z) \to 1/2$ as $z \to \infty$).

A ratio of polynomials is a *rational function* and is uniquely determined by its poles, zeros and a constant gain factor:

$$f(z) = \frac{a_n z^n + \cdots + a_1 z + a_0}{b_m z^m + \cdots + b_1 z + b_0} = K\frac{(z-z_1)\cdots(z-z_n)}{(z-p_1)\cdots(z-p_m)}$$

If we count poles or zeros at ∞ (corresponding to the cases $n > m$ or $n < m$, respectively), and count multiplicities, then the total number of poles = the total number of zeros = $\max(n, m)$.

Power Series and Laurent Series

A power series:

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_o)^n$$

is *absolutely convergent* in a region if $\sum_{n=0}^{\infty} |c_n| |z - z_o|^n < \infty$ at all points z in the region. f is analytic in the region of absolute convergence, and conversely if f is analytic at z_o then it has such a power series representation.[2]

In particular, if f is analytic at z_o (f' exists) then *all* derivatives $f^{(n)} = \frac{d^n f}{dz^n}$ exist at z_o!! In fact, $c_n = \frac{1}{n!} f^{(n)}(z_o)$.

A real-valued function $f(x)$ is called analytic if it is equal to such a power series. Note that even if $f^{(n)}$ exists for all n it may not be analytic! For example, $f(x) = e^{-1/x^2}$, $x \neq 0$, $f(0) = 0$. Then $f^{(n)} = (\text{polynomial in } x^{-1}) \times e^{-1/x^2} \to 0$ as $x \to 0$, so $f^{(n)}(0) = 0$ for all n. This suggests a power series with all terms 0, but $f(x) \neq 0$!!! Hence, f is not analytic at 0 even though it has infinitely many derivatives.

If $f(x)$ is a real analytic function then it can be extended to $f(z)$ as an analytic function; that is, the power series in x will work with complex z. Note that e^{-1/z^2} in fact has an essential singularity at $z = 0$ (if $z = jy \to 0$, $e^{-1/z^2} \to \infty$, but if $z = x \to 0$, $e^{-1/z^2} \to 0$).

More generally, a *Laurent series* is a power series with both positive and negative powers:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_o)^n$$

It is analytic in the region of convergence (ROC), which is z for which $\sum_{n=-\infty}^{\infty} |c_n| |z - z_o|^n < \infty$. The ROC, if it exists, has the form of an annular ring: $R_1 < |z| < R_2$.

[2] The term *holomorphic* is often used interchangeably with *analytic.* In fact, holomorphic means specifically a function of a complex variable that is differentiable, and analytic is more generally any function that can be represented by an absolutely convergent power series (e.g., real valued or matrix valued analytic functions). A basic theorem in complex analysis is that a complex function is holomorhpic iff it is analytic.

Example 13 $c_n = e^{n^2}$*: the series does not converge for any* z

Example 14 $f(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ *for* $|z| < 1$*;* $f(z) = \sum_{n=-\infty}^{-1} -z^n = \frac{1}{1-z}$ *for* $|z| > 1$*. To see the second case:*

$$\sum_{n=-\infty}^{-1} -z^n = -z^{-1} \sum_{m=0}^{\infty} z^{-m} = \frac{-z^{-1}}{1-z^{-1}} = \frac{1}{1-z}, \textit{ for } \left|z^{-1}\right| < 1$$

Hence, the function $\frac{1}{1-z}$ *has two different Laurent series corresponding to two distinct ROCs. Note that the ROCs are bounded by the circle* $|z| = 1$ *which passes through the pole at* $z = 1$*.*

In general if $f(z)$ has various poles, it has several different Laurent series representations with ROCs bounded by circles passing through the poles. Two Laurent series for the same function cannot have partially overlapping ROCs: either the ROCs do not overlap at all, or the two series have the exact same ROC and in fact are the same series (have identical coefficients).

Contour Integration

Let Γ be an arc or contour, not necessarily closed, with f analytic on Γ. Then the *contour integral* of f on Γ is:

$$\int_{\Gamma} f(z)\,dz = \lim_{\substack{|\Delta z_i| \to 0 \\ N \to \infty}} \sum_{i=1}^{N} f(z_i)\,\Delta z_i$$

where z_i lie on Γ and $\Delta z_i = z_i - z_{i-1}$. This contrasts with the *line integral* against an arclength parameter, $\int_{\Gamma} f(x,y)\,ds$ in that the weight function is $ds = |dz|$, i.e. the differential arclength $\Delta s = |\Delta z|$.

If $\Gamma = C$ is a simple closed curve, we take the direction of integration to be counterclockwise (ccw) by default, and denote it as:

$$\oint_C f(z)\,dz$$

Clockwise integration can be replaced by minus the counterclockwise integral.

Theorem 15 f *is analytic in a region (without holes) iff* $\oint_C f(z)\,dz = 0$ *for all simple closed curves* C *in the region.*

In fact, the Cauchy-Riemann equations, on the one hand, and this integral theorem, on the other hand, can be shown to be equivalent via Green's Theorem.

If f is analytic on a simple closed curve C, but not necessarily inside, then $\oint_C f(z)\,dz$ is not necessarily zero. The *residue* of f about C is:

$$\underset{C}{res} f = \frac{1}{2\pi j} \oint_C f(z)\,dz$$

If f has a singularity at z_0, then the residue of f at z_0 is:

$$\underset{z_0}{res} f = \frac{1}{2\pi j} \oint_C f(z)\,dz$$

where C is a circle centered about z_0 of sufficiently small radius such that f is analytic in and on C except for the singularity at z_0.

Theorem 16 *If C can be continuously deformed to C' without passing through any singularities of f, then $\oint_C f dz = \oint_{C'} f dz$. If f is analytic on C and inside except for singularities at z_1, z_2, $\cdots$, z_N, then:*

$$\frac{1}{2\pi j}\oint_C f(z)\,dz = \sum_{i=1}^{N} \underset{z_i}{res} f$$

Lemma 17 *If C is a simple closed curve containing z_o:*

$$\frac{1}{2\pi j}\oint_C \frac{1}{(z-z_o)^{n+1}} dz = \delta[n] = \begin{cases} 1, & n=0 \\ 0, & otherwise \end{cases}$$

Proof Without loss of generality, let C be a circle of radius R about z_o: $z = z_o + R\,e^{j\theta}$, $\theta : 0 \to 2\pi$, $dz = jR\,e^{j\theta}d\theta$. Therefore:

$$\frac{1}{2\pi j}\oint_C \frac{dz}{(z-z_o)^{n+1}} = \frac{1}{2\pi j}\int_{\theta=0}^{2\pi} \frac{jR\,e^{j\theta}d\theta}{R^{n+1}e^{j(n+1)\theta}} = \frac{R^{-n}}{2\pi}\int_{\theta=0}^{2\pi} e^{-jn\theta}d\theta = \delta(n)$$

Theorem 18 (Cauchy Integral Formula) *If $f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_o)^n$ and C is a simple closed curve encircling z_o within the ROC of the series then:*

$$c_n = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_o)^{n+1}} dz$$

As a special case, if f is analytic in and on C and z_o is inside C then:

$$f(z_o) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_o} dz$$

and for $n = 1, 2, 3, \cdots$:

$$f^{(n)}(z_o) = \frac{n!}{2\pi j}\oint_C \frac{f(z)}{(z-z_o)^{n+1}} dz$$

Corollary 19 *Let $f(z)$ has an m^{th} order pole at z_o, that is $f(z) = \frac{g(z)}{(z-z_o)^m}$ where g is analytic at z_o. If z_o is a simple pole ($m = 1$) then:*

$$\underset{z_o}{res} f = g(z_o)$$

and if $m > 1$ then:

$$\underset{z_o}{res} f = \frac{1}{(m-1)!} g^{(m-1)}(z_o)$$

Thus, we can evaluate integrals by computing residues.

Maximum Principle and Other Properties

There are restrictions on the locations of extrema of analytic functions. The basic result is:

Theorem 20 (Maximum Principle) *Let f be analytic in and on C. Then* $\max_{z \text{ in } C} |f(z)|$ *occurs on C. If the maximum value is also achieved at a point inside C, then f is constant.*

The following result can be derived from the maximum principle.

Corollary 21 (Rouche's Theorem) *If f and g are analytic inside and on C, and $|f| > |g|$ on C* (strict!), *then f and $f+g$ have the same number of zeros (counting multiplicities) inside C.*

Example 22 $z^6 + 3z^2 + 1$*: let* $f = 3z^2$, $g = z^6 + 1$. *For* $|z| = 1$, $|f| = 3$, $|g| \leq 1 + 1 = 2$, *so* $|f| > |g|$. *Thus, f and $f + g$ have the same number of zeros in $|z| < 1$. That is, $z^6 + 3z^2 + 1$ has exactly two zeros in $|z| < 1$, and therefore 4 zeros in $|z| > 1$. For $z^6 + 2z^2 + 1$, this test fails: $|2z^2| \geq |z^6 + 1|$ for $|z| = 1$, namely at $z = 1$ these two terms are equal.*

Let C be a simple closed curve, f analytic on C with no poles or zeros on C. As z winds around C once (in the ccw direction), let $w = f(z)$ trace out a curve Γ in the w-plane. The *winding number* N of Γ is the number of times Γ encircles the origin; $N > 0$ if ccw, $N < 0$ if cw (clockwise).

Theorem 23 (Cauchy's Principle of the Argument) *If f is analytic on C with no zeros or poles on C, having winding number N, Z zeros in C and P poles in C, then:*

$$N = Z - P$$

Proof. Let $w = f(z)$ trace out a curve Γ in the w-plane as z winds around C once in the ccw direction. We first show:

$$N = \frac{1}{2\pi} \oint_{\Gamma} d \arg w = \frac{1}{2\pi j} \oint_{\Gamma} d \log w = \frac{1}{2\pi j} \oint_{\Gamma} \frac{dw}{w}$$

Note $\log w = \ln |w| + j \arg(w)$ and Γ closes on itself; as Γ winds around the origin, $\ln |w|$ stays single valued so $\sum \Delta \ln |w| = 0$, whereas the argument increments by 2π for each ccw encirclement of the origin, and by -2π if cw. That is, as w winds around the origin N times, $\sum \Delta \arg w = 2\pi N$. Thus:

$$N = \frac{1}{2\pi j} \oint_{C} d \log f(z) = \frac{1}{2\pi j} \oint_{C} \frac{f'(z)}{f(z)} dz$$

Now write $f(z)$ as:

$$f(z) = g(z) \prod (z - z_i)^{m_i} / \prod (z - p_i)^{n_i}$$

where g is analytic with neither poles nor zeros inside or on C, and z_i, p_i are the zeros and poles, respectively, of f in C. Then:

$$\log f = \log g + \sum m_i \log(z - z_i) - \sum n_i \log(z - p_i)$$

Now:

$$\frac{1}{2\pi j}\oint_C d\log g(z) = \frac{1}{2\pi j}\oint_C \frac{g'(z)}{g(z)}dz = 0$$

since g/g' has no poles in C (its poles would be poles of g or zeros of g). Also, if z_o is inside C:

$$\frac{1}{2\pi j}\oint_C d\log(z-z_o) = \frac{1}{2\pi}\oint_C d\arg(z-z_o) = 1$$

Hence:

$$N = \frac{1}{2\pi j}\oint_C d\log f(z) = 0 + \sum m_i - \sum n_i = Z - P$$

■

The Cauchy Principle of the Argument is the basis of the Nyquist diagram in control theory.

Conformal Mapping Properties

If we view a complex function $w = f(z)$ as a mapping of points, curves and regions in the z-plane to points, curves and regions in the w-plane, it is called a *conformal mapping.* A conformal mapping preserves oriented angles (if two curves intersect at some angle in the z-plane, the the corresponding curves in the w-plane intersect at the same angle). It turns out a mapping is conformal iff it is representable by an analytic function.

Here, we are concerned with two issues in conformal mapping: the Schwarz Reflection Principle, and the special class of maps called bilinear transforms.

We first define the notion of a *symmetric point.* The general definition is beyond the scope of these notes. The symmetric point of the complex number z with respect to a line is the complex point that is the reflection of z in the line. The symmetric point is on the other side of, and equidistant from, the line. The symmetric point of the complex number z with respect to a circle is the complex point along the same ray emanating from the center of the circle, and such that the geometric mean of the distances from the center is the radius of the circle. The notion of a symmetric point can be generalized to a wide variety of curves. If z lies *on* the curve C, then its symmetric point with respect to C is itself. We shall only require the following three cases:

Definition 24 *The following are three special cases of the definition of a* symmetric point:

1. *The symmetric point of s with respect to the real axis is s^*.*

2. *The symmetric point of s with respect to the imaginary axis is $-s^*$.*

3. *The symmetric point of z with respect to the unit circle is $1/z^*$. In particular, 0 and the point at ∞ form a symmetric pair.*

The basic result from complex analysis that we shall rely on is:

Theorem 25 (Schwarz Reflection Principle) *If an analytic function maps the curve C onto the curve Γ, then it maps symmetric points of C to symmetric points of Γ.*

Example 26 *If $H(s)$ is a rational function that is real for real s, then $H(s) = H^*(s^*)$ for all complex s. Now, $H^*(s^*)$ is a ratio of polynomials in s whose coefficients are the conjugates of the coefficients of $H(s)$. Hence, all coefficients of $H(s)$ must be real, and the poles and zeros occur in complex conjugate pairs. Note that here we have generalized the relation $H(\sigma) = H^*(\sigma)$ for real σ to the case of complex σ. Viewed another way, if we define $R(\sigma) = Re(H(\sigma)) = (H(\sigma) + H^*(\sigma))/2$, for real $s = \sigma$, then the analytic continuation of R to the complex plane is $R(s) = (H(s) + H^*(s^*))/2$. Indeed, in general, $H^*(s)$ is* not *an analytic function (the conjugate function $f(s) = s^*$ is not analytic). But the double conjugation in $H^*(s^*)$ has the effect of conjugating the coefficients of H while leaving s unconjugated, so that $H^*(s^*)$ is in fact analytic.*

A mapping of the form:

$$w = \frac{az + b}{cz + d} \tag{1}$$

is called a *linear fractional transform,* a *bilinear transform* or a *Möbius transform.* If $ad - bc = 0$, the mapping is degenerate, reducing to $w = b/d$ constant. Therefore, we specify that $ad - bc \neq 0$. When $cz + d = 0$, we define $w = \infty$; and when $z = \infty$ we define $w = a/c$. Note that $ad - bc = 0$ prevents $az + b = 0$ and $cz + d = 0$ at the same point z.

Theorem 27 *A bilinear transform $w = f(z)$ given by (1) satisfies the following properties:*

1. *If $w = f(z)$ is defined via $\{a, b, c, d\}$ and $v = g(w)$ is defined via $\{a', b', c', d'\}$, then $v = g(f(z))$ is a bilinear transform specified by $\{a'', b'', c'', d''\}$ where:*

$$\begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2}$$

2. *It is a one-to-one mapping of the closed complex plane onto itself (the closed complex plane includes the point at infinity); thus, for each z, $w = \frac{az+b}{cz+d}$ exists, and for each w, there is a unique point $z = \frac{a'w+b'}{c'w+d'}$ where the coefficients $\{a', b', c', d'\}$ are found as the elements of the inverse of the matrix formed by $\{a, b, c, d\}$; note $ad - bc \neq 0$ ensures this matrix is invertible.*

3. *A circle or a line in the z-plane maps to a circle or a line in the w-plane. Here, a line can be thought of as a circle which contains the point at ∞, so a line or circle is called a* generalized circle.

4. *Consider a curve C that cuts the z-plane into disjoint regions Ω_1, Ω_2; for example, a line has a left side and a right side, and a simple closed curve has an inside and an outside; then a bilinear transform $w = f(z)$ maps C to a curve Γ in the w-plane which cuts the plane into disjoint regions Ω_1' and Ω_2', and moreover either the entire region Ω_1 maps to the entire region Ω_1' and Ω_2 to Ω_2', or Ω_1 maps to Ω_2' and Ω_2 to Ω_1'.*

5. *The region mapping discussed in 4 above obeys the following rule. If we trace out C in a certain direction and then one region, say Ω_1, is to the left of the curve. As z follows C then w follows Γ in a certain direction, and hence one region, say Ω_1' is to the left of Γ. Then Ω_1 maps to Ω_1'. For example, if C is a circle, then its inside is on the left*

as C is traced out in the counterclockwise direction. Suppose Γ is a circle, too. Then the inside maps to the inside or to the outside depending on whether Γ is traced out counterclockwise or clockwise, respectively.

6. *The bilinear transform is uniquely determined by selecting any three points in the z-plane and specifying where they map to in the w-plane.*

Proposition 28 *The most general analytic one-to-one mapping of the unit circle onto itself is:*

$$w = e^{j\theta} \frac{z - k}{-k^* z + 1} \tag{3}$$

where $\theta \in \mathbb{R}$ and $|k| \neq 1$. Moreover, if $|k| < 1$, this maps $\{|z| \leq 1\}$ onto $\{|w| \leq 1\}$, and if $|k| > 1$ this maps $\{|z| \leq 1\}$ onto $\{|w| \geq 1\}$. The inverse map is:

$$z = \frac{e^{-j\theta} w + k}{k^* e^{-j\theta} w + 1} \tag{4}$$

If $k = 0$, then $w = e^{j\theta} z$ is a special case, and if $k = \infty$, then $w = e^{j\theta} \frac{1}{z}$ is another special case. The most general analytic one-to-one mapping of the unit circle onto the imaginary axis is:

$$w = \sigma \frac{\gamma^* z + 1}{z - \gamma} \tag{5}$$

where $\sigma \in \mathbb{R} \backslash \{0\}$ and $\mathrm{Re}(\gamma) \neq 0$. Moreover, if $\sigma > 0$, then if $\mathrm{Re}(\gamma) > 0$ this maps $\{|z| \leq 1\}$ onto the LHP $\{\mathrm{Re}(w) \leq 0\}$ and if $\mathrm{Re}(\gamma) < 0$ this maps $\{|z| \leq 1\}$ onto the RHP $\{\mathrm{Re}(w) \geq 0\}$. If $\sigma < 0$, these conditions are reversed. The inverse map is:

$$z = \frac{w + \sigma\gamma}{w - \sigma\gamma^*} \tag{6}$$

This represents the most general analytic one-to-one mapping of the imaginary axis onto the unit circle.

Corollary 29 *The most general rational functions that map the unit circle onto itself, unit circle to imaginary axis, or imaginary axis to unit circle are products of the respective first-order functions in the above proposition.*

We recognize mappings of the unit circle onto itself as digital all-pass transfer functions, and mapping of the imaginary axis to the unit circle as analog all-pass transfer functions. The Schwarz Reflection Principle can similarly be used to characterize real (or zero-phase) analog and digital transfer functions (i.e., transfers functions that are real valued in the frequency domain).