SIGNALS

Note Title 1/15/2006

Signals:
- entities that represent or contain information
- real or complex valued functions considered here
  - deterministic & stochastic (random)
  - generalizations include vector-valued signals (e.g., electromagnetic fields)

- our signals populate a signal space and will be viewed as vectors
  - sum vectors
  - multiply by scalars
  - linear independence
  - orthogonality
  - distance
  - projection

Systems:
- process or operate on signals
- mathematically, systems are operators acting on signals which are vectors or functions

Notations:

$\mathbb{R}$ real

$\mathbb{Z}$ integer

$\mathbb{Q}$ rational

$\mathbb{C}$ complex

$\mathbb{N}$ natural #'s $(1, 2, \ldots)$

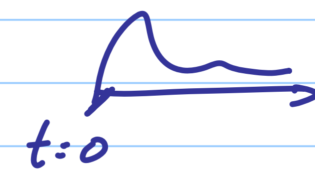
$\mathbb{R}^n$ n-dim. real vectors

(similarly for $\mathbb{Z}^n$, $\mathbb{C}^n$, etc.)

Classification of signals

continuous-time $\quad t: -\infty \to \infty \quad (t \in \mathbb{R})$

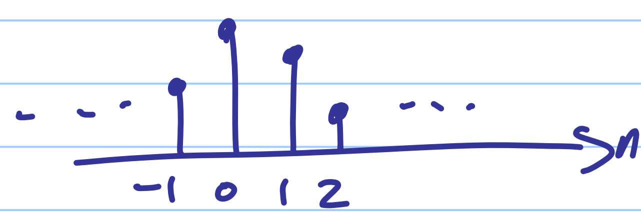
in some cases start at $t=0$:



discrete-time: $\quad t \in$ discrete set, say $\{t_1, t_2, t_3, \dots\}$

$s(t_1), s(t_2), s(t_3), \dots$

Generally take time to be $n \in \mathbb{Z}$: $\quad s(n) \quad (n \to -\infty \;\&\; n \to +\infty)$



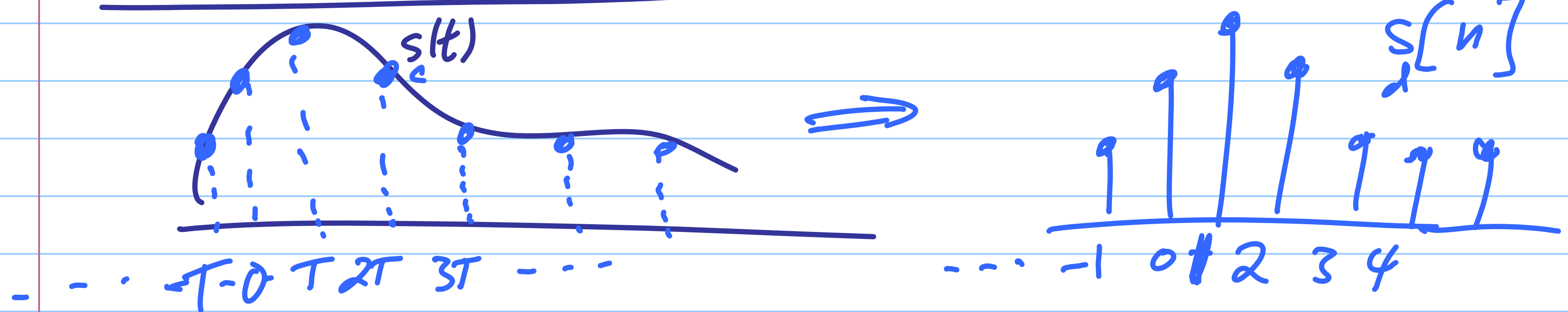
"stem plot"

Can obtain discrete-time from continuous-time by sampling:

$$s_d[n] = s_c(nT)$$

discrete ↗ (d)   ↖ cont (c)

, often use $s[n]$ to indicate <u>discrete</u>

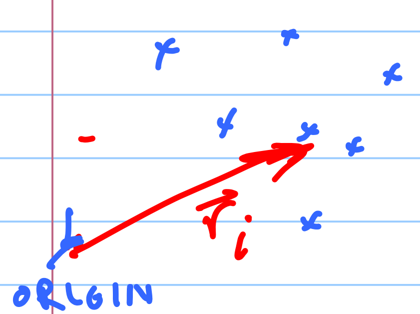


T = sample period (sec) $\qquad f_s = \frac{1}{T}$ = sampling rate (Hz)

$\omega_s = 2\pi f_s = 2\pi/T$ = sampling rate (rad/sec) multidimensional: $s(x,y)$

ex: $s(\vec{r})$, $\vec{r} \in \mathbb{R}^2$ or $\mathbb{R}^3$ spatial point

a field or continuous image



sensor array, sensors at points $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots$

Measure a discrete multidim. signal $s(\vec{r}_i)$

__multichannel:__ a signal with multiple components

~vector-valued: $\vec{s}(t) = \begin{bmatrix} s_1(t) \\ s_2(t) \\ \vdots \\ s_N(t) \end{bmatrix}$

__ex:__ T-1 phone line = 25 signals (24 voice + 1 control)

__ex:__ complex valued signal can be viewed as 2-channel real:

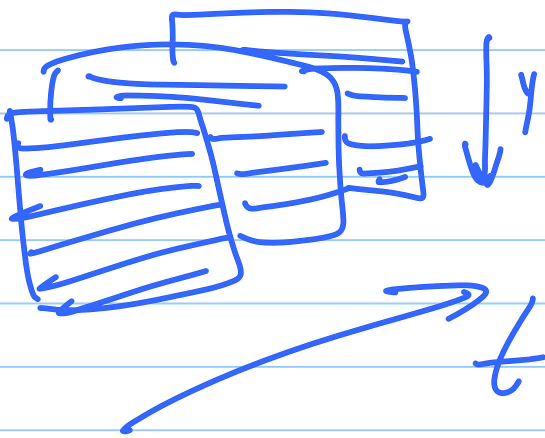
$$z(t) = x(t) + j\,y(t) \sim \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

ex: color video signal: 3channel, 3-dim. signal

$(r(x,y,t), g(x,y,t), b(x,y,t)) \sim RGB$: Can map to $YCrCb$ or YIQ (NTSC)

<u>analog video vs. digital video:</u> y & t discrete for both

horizontal lines → y frames or fields → t



Difference is whether x is continuous or discrete

Y = luminance = weighted sum of RGB (not equal)

~ perceived as colorless gray scale

$C = Cr + jCb$ = chrominance

(Y, C) is a 2-channel, 3-dim signal with one real, one cplx. ch.

<u>Quantization</u>: a signal is quantized when its <u>values</u> come from a discrete set, called the <u>quantization levels</u>

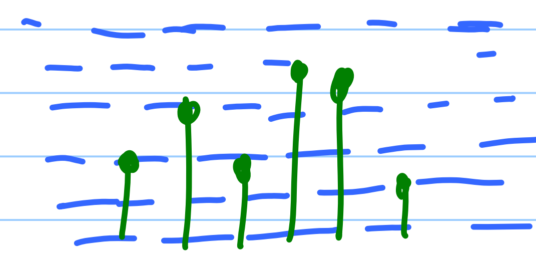
digital $\equiv$ discrete-time & quantized

<u>analog</u> $\equiv$ continuous-time & continuous-valued

Some cont-time sys are not "true" analog, eg (idealized) pulsed systems

When studying "digital" sig/sys, we must often <u>ignore</u> quant. & so are really looking at disc-time, not digital

<u>ex</u>: 8-bit values ~ 256 quant levels



Black box concept of a system

$$S_{IN} \longrightarrow \boxed{\text{System}} \longrightarrow S_{OUT}$$

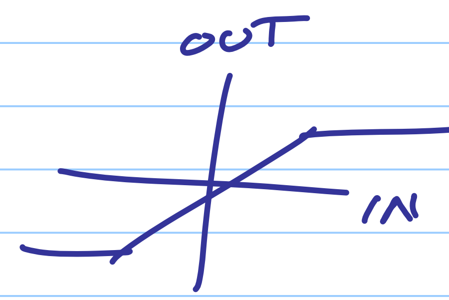
- describe input-output behavior, internal details do not matter
- **realization:** given I-O behavior, implement the system

eg: $y = (2a) + (2b)$ vs. $y = 2(a+b)$

Note: one sys requires 2 mult, 1 add, the other one mult, one add

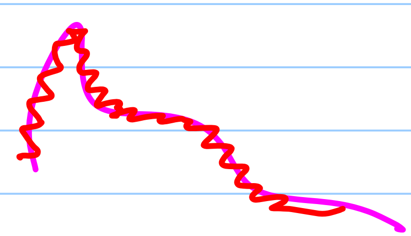
Insides <u>do</u> matter, ex: quantization effects ~ nonlinearity
cost vs. performance

<u>ex:</u> analog ckts. often exhibit <u>saturation</u>
in dig. sys., this occurs as <u>overflow</u>



<u>ex:</u> dig. sys. introduce <u>roundoff error</u>
analog sys. introduce <u>thermal noise</u>
(due to random motion of electrons)
similar characteristics

<u>Sensitivity:</u> dependence on component values
- analog: manuf never exact
- dig.: coeff. stored with finite precision



<u>Remark:</u> In communications (ECE102), will study information theory & related ideas such as lossy/lossless compression

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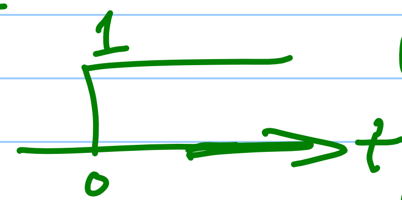
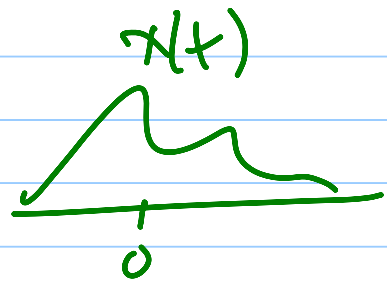
<u>TWO SPECIAL SIGNALS</u>

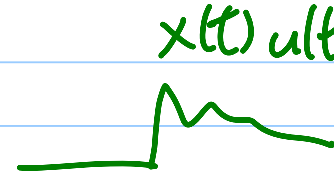
unit step: continuous time



$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$

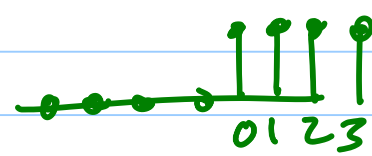
undefined @ $t=0$ (irrelevant)





$x(t) \cdot u(t) = \begin{cases} x(t) & t > 0 \\ 0 & t < 0 \end{cases}$

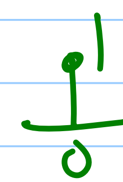
discrete-time:



$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n \leq -1 \end{cases} \Leftarrow u[0] = 1$

$x(n) \cdot u(n) = \begin{cases} x(n) & n \geq 0 \\ 0 & n \leq -1 \end{cases}$

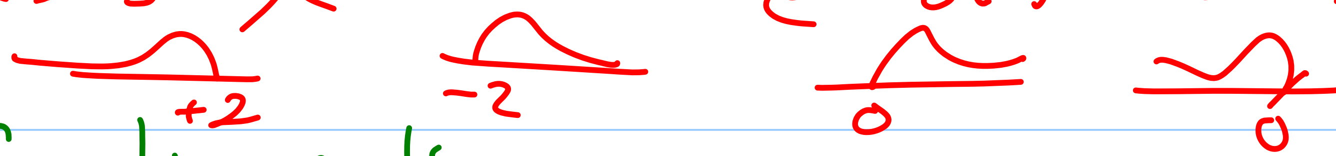
discrete <u>impulse</u>: $\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$



Analog impulse is BIZARRE, will study later

$s([-t]+2) \leftarrow s(t+2) \leftarrow s(t) \rightarrow s(-t) \rightarrow s(-[t-2])$

$= s(-t+2)$



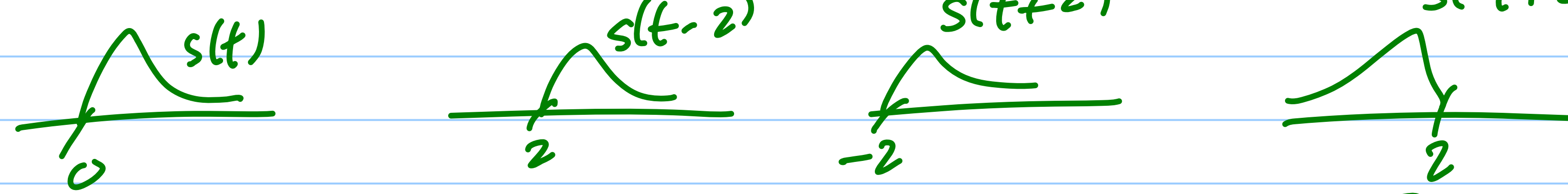
Graphing signals:

$s(t-t_0) \quad s(t+t_0) \quad s[n-n_0] \quad s[n+n_0] \quad s[-n] \quad s[-n+2]$

TRICKS: · locate the "0" pt.

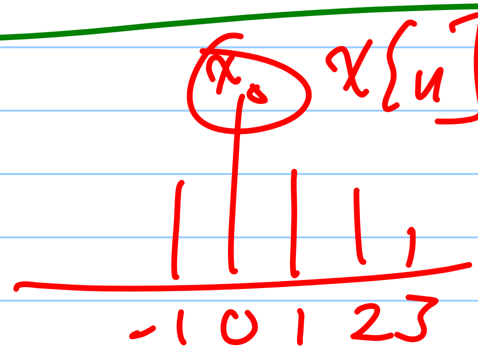
· flip if there is a $-t$ or $-n$

ex.:

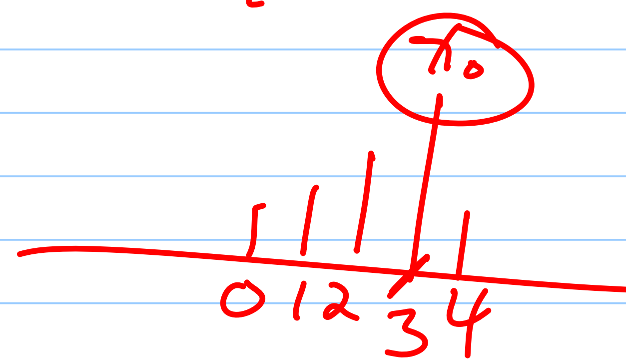


when $t=2$, $-t+2=0$

$x[n]$

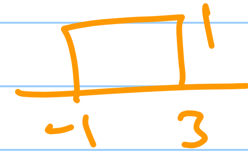


$x[-n+3]$



Support of a signal

examples: ① graph $u(-t+1)+2u(t-2)$

② express  in terms of $u(t)$

③ graph $u[-n+1]+2\delta[n-2]+3u[n-2]$

THEOREM

$$x[n]=\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$

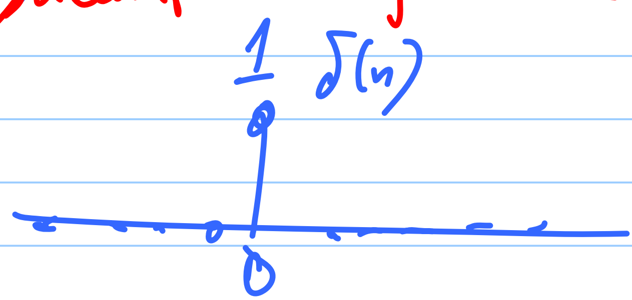
superposition of impulses

key idea

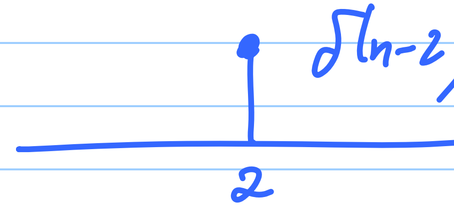
$x_k\,\delta(n-k)$

→ decompose a signal as a linear combination of "elementary" signals

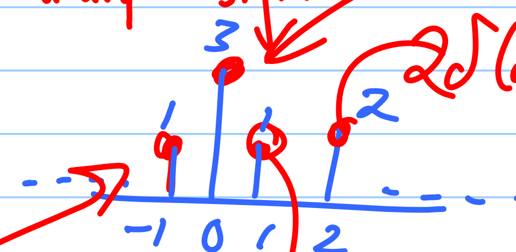
$\delta(n)$



$\delta(n-2)$



$3\delta(n)$ $2\delta(n-2)$



$\delta(n+1)$ $\delta(n-1)$

Sine waves

$$\boxed{j \equiv \sqrt{-1}} \quad (\text{not } i)$$

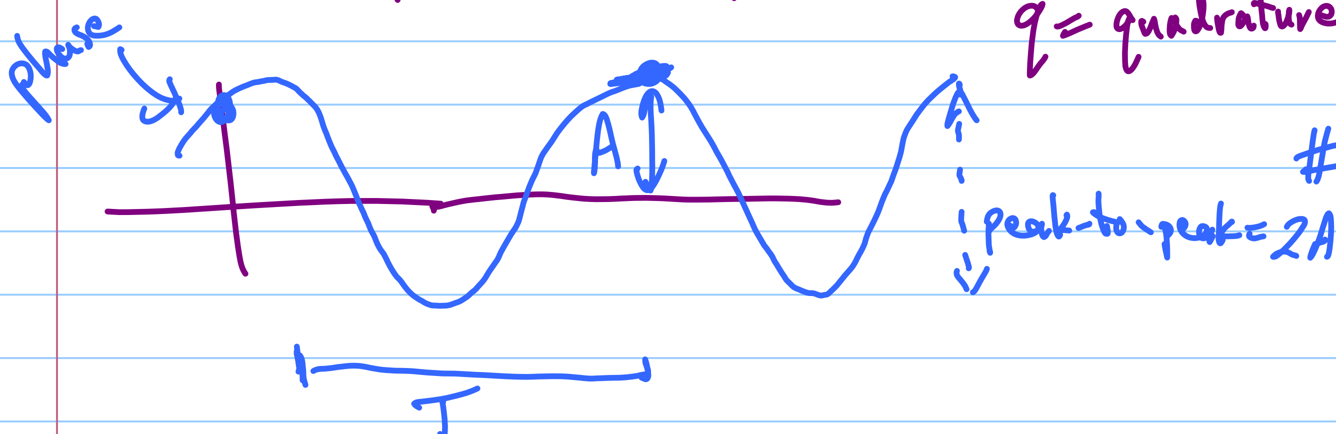
$$v(t) = A\cos(2\pi f_0 t + \theta) = i\cos 2\pi f_0 t - q\sin 2\pi f_0 t$$

Communication jargon: f_0 = carrier frequency (Hz)

$\omega_0 = 2\pi f_0 \sim$ rad/sec $\qquad T = \frac{1}{f_0}$ sec. = period

A = amplitude $\quad \theta$ = phase $\qquad i$ = in-phase component

q = quadrature component



cycles per sec (cps) $\equiv$ Hertz

in-phase

Reference phase to <u>cosine carrier</u>, ie $\cos(2\pi f_0 t) \sim 0°$

The <u>quadrature</u> carrier is $\sin(2\pi f_0 t)$

<u>TRIG:</u> $\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

very useful: $\cos A \cos B = \frac{1}{2}\cos(A+B) + \frac{1}{2}\cos(A-B)$

sum & difference

<u>eg</u>: $\cos(2\pi f_1 t) \cdot \cos(2\pi f_2 t) = \frac{1}{2}\cos(2\pi(f_1+f_2)t) + \frac{1}{2}\cos(2\pi(f_1-f_2)t)$

$\therefore$ when <u>multiply</u> sinewaves, get SUM & DIFFERENCE frequencies

For now, we consider ONE FIXED FREQUENCY

$\Downarrow$

PHASORS

(more general case requires more powerful tools, e.g. Fourier analysis)

$v(t) = A\cos(2\pi f_0 t + \theta)$ $\qquad\qquad$ $i = A\cos\theta$

Write $\quad V = Ae^{j\theta} = i + jq$ $\qquad\qquad$ $q = A\sin\theta$

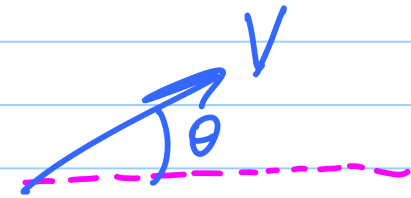
$$v(t) = \mathrm{Re}\left(Ve^{j2\pi f_0 t}\right)$$ $\qquad$ $V =$ <u>phasor</u>

<u>ex</u>: $\quad 3\cos 2\pi f_0 t - 4\sin 2\pi f_0 t \Longrightarrow V = 3+j4$

$$v(t) = \mathrm{Re}\left((3+j4)(\cos 2\pi f_0 t + j\sin 2\pi f_0 t)\right) = 3\cos 2\pi f_0 t - 4\sin 2\pi f_0 t$$

CAREFUL!

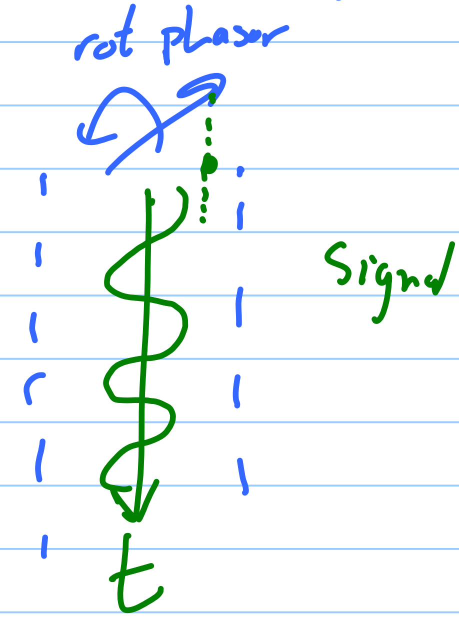
static view of a phasor



<ref. carrier>

<u>rotating phasor</u>:

rotates with angular velocity ω rad/sec.



phasor analysis assumes a <u>common</u> f_0

superposition of sinewaves @ same f_0 follows complex addition rule

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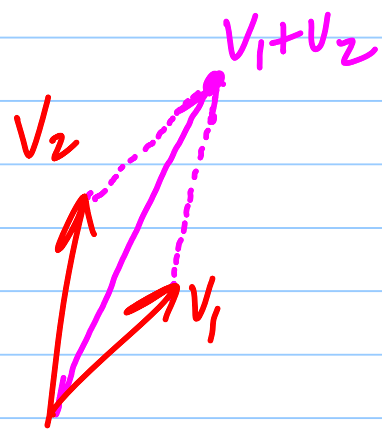
if $v_1(t) \sim V_1$ $\quad v_2(t) \sim V_2$ $\quad$ then $v_1(t) + v_2(t) \sim V_1 + V_2$



(parallelogram law)

<u>ie</u>: $|V_1|\cos(\omega_0 t+\theta_1) + |V_2|\cos(\omega_0 t+\theta_2)$

$= |V|\cos(\omega_0 t+\theta)$

where $V = |V|e^{j\theta} = |V_1|e^{j\theta_1} + |V_2|e^{j\theta_2}$

(easiest seen in rect. form: $(i_1\cos - q_1\sin) + (i_2\cos - q_2\sin)$

$= (i_1+i_2)\cos - (q_1+q_2)\sin$

Notation: θ in rad $\sim e^{j\theta}$ $\qquad$ θ in deg $\sim \angle\theta$

ex: $3\angle 30^\circ = 3e^{j\pi/6}$

NEVER mix it up: e^{j30° is **BAD**

$\Rightarrow$ Can use phasors to study **ckts** (& other systems governed by diff eqns) based on:

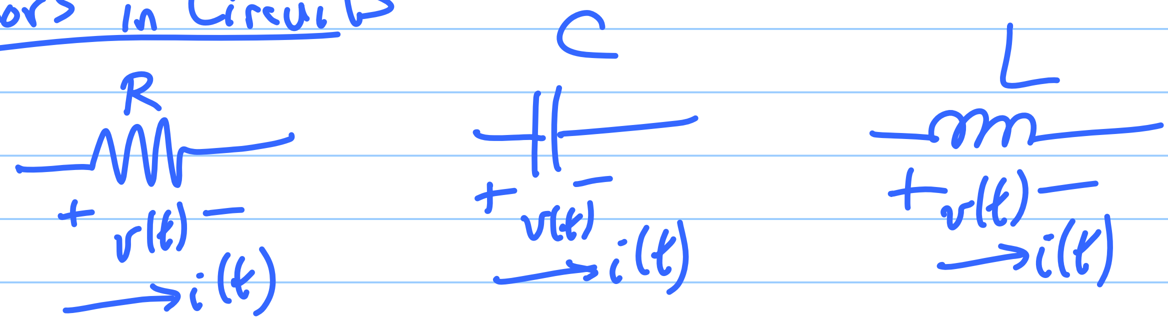
$v'(t) \Leftrightarrow j\omega V$: $\qquad v(t) = \mathrm{Re}(Ve^{j\omega t})$

$v'(t) = \mathrm{Re}(j\omega V e^{j\omega t})$

$\frac{d}{dt} \Rightarrow$ MULTIPLY BY $j\omega$ IN PHASOR DOMAIN

$\int dt \Rightarrow$ MULT BY $\frac{1}{j\omega}$ in PHASOR DOM.

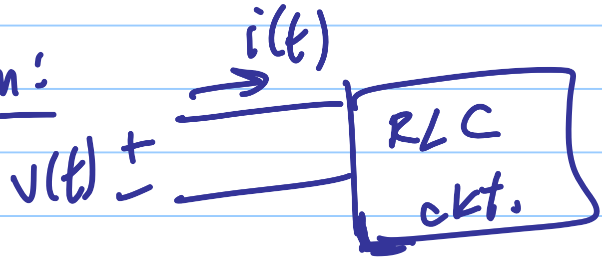
Phasors in Circuits



$$\boxed{v = Ri \qquad v = L\frac{di}{dt} \qquad i = C\frac{dv}{dt}}$$ Time Domain

Phasor form: $V = RI$ $\qquad V = j\omega L I$ $\qquad V = \frac{1}{j\omega C} I$

Thm:



V & I related via complex constant (for fixed ω)

$V = ZI$

$I = YV$

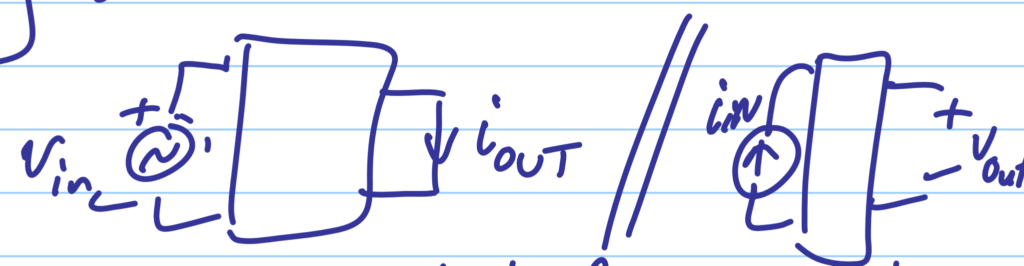
$Z = \frac{1}{Y}$ (complex division)

meaning:

any voltage (or current) in the ckt

source $A_s \cos(\omega t + \theta_s)$ @ freq ω

Ex:



All volt & curr in ckt are sinewaves @ that frequency!

Relate phasor for "input" to phasor of "output" (V or I)

$V = Z I$ $\qquad$ $Z = \frac{1}{Y}$ $\qquad$ $Z = R + jX$

$I = Y V$ $\qquad\qquad\qquad$ $Y = G + jB$

R: resistance $\quad$ X: reactance $\quad$ G: conductance $\quad$ B: susceptance

Y: admittance $\quad$ Z: impedance

R, X, Z units: Ω $\qquad$ G, B, Y units: Ω^{-1} (siemens or mho)

$\mho$

(Z or Y called immittance)

eg:

3Ω $\quad 0.4H$ @ $\omega_0 = 10$ rad/sec

i

$+v_1-$ $\quad +v_2-$

$$+v_{TOT}- = v_1 + v_2 = 3i + 0.4\frac{di}{dt} \quad (i = \text{same})$$

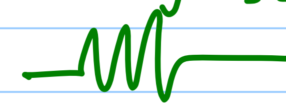
$$V_{TOT} = (3 + 0.4 \times j10) I$$

$$\therefore \boxed{Z = 3 + j4\ \Omega}$$

3 $\quad$ j4

$\Rightarrow Z = 3 + j4$

$3+j4\ \Omega$



$Z_{TOT} = Z_1 + Z_2$

impedances in series add (like resistors)

Use  symbol for generic impedance

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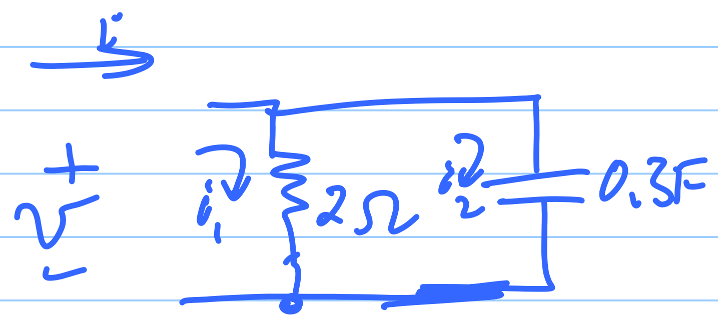
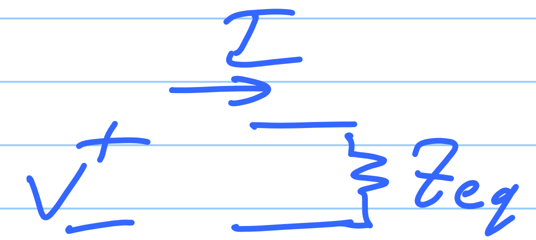


@ 20 rad/sec



$$\left.\begin{array}{l} i = i_1 + i_2 \\ v = v_1 = v_2 \end{array}\right\} \quad i = \frac{1}{2} v + 0.3 \frac{dv}{dt}$$

$$I = \frac{1}{2} V + j\,0.3 \times 20 V = (0.5 + j\,6) V$$

$$\therefore Y = 0.5 + j6 \ \Omega^{-1} \qquad \left(Z = \frac{0.5 - j6}{(0.5)^2 + 6^2} \ \Omega \right)$$

admittances in parallel add:

$Y = Y_1 + Y_2$ $\longrightarrow$ 

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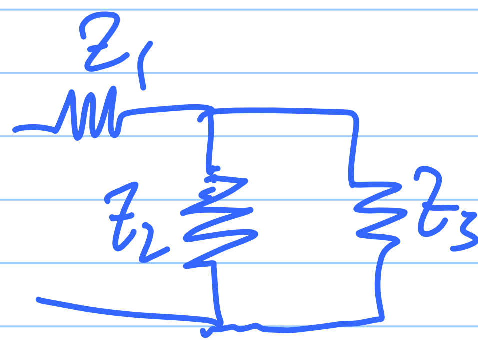
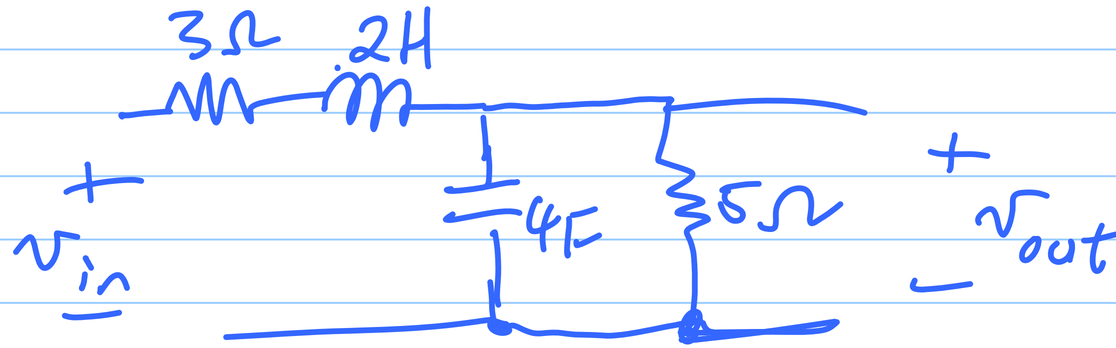
ex:

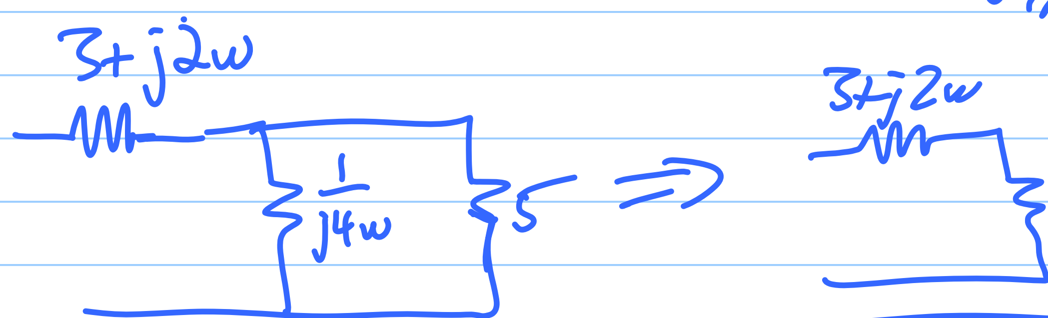


$$Z_{eq} = Z_1 + \frac{Z_2 \cdot Z_3}{Z_2 + Z_3}$$

etc.



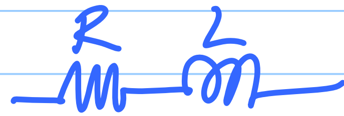
If v_{in} sinewave @ ω, find $\frac{V_{OUT}}{V_{IN}}$ as a fn. of ω:



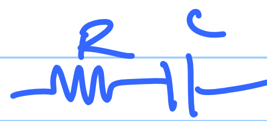
$$\frac{5 \cdot \frac{1}{j4\omega}}{5+\frac{1}{j4\omega}} = \frac{5}{1+j20\omega}$$

Voltage divider:

$$\boxed{\frac{V_{OUT}}{V_{IN}} = \frac{\frac{5}{1+j20\omega}}{\frac{5}{1+j20\omega} + 3+j2\omega}}$$

Remark: 

$\Rightarrow Z = R + j\omega L$: $X_L = \omega L > 0$

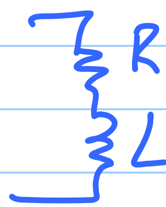


$\Rightarrow Z = R + \frac{1}{j\omega C} = R - j\frac{1}{\omega C}$ $\therefore X_c = -\frac{1}{\omega C} < 0$

inductive reactance $X > 0$ / capacitive reactance $X < 0$

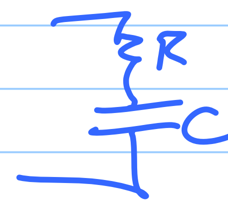
susceptance $B < 0$ / susceptance $B > 0$


$Z = R_{eff} + \begin{cases} j\omega L_{eff} & \text{if } X > 0 \\ \frac{-j}{\omega C_{eff}} & \text{if } X < 0 \end{cases}$

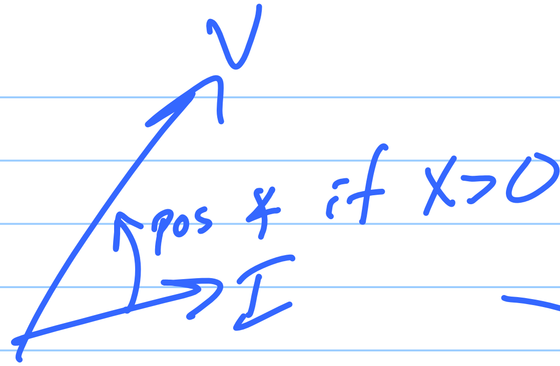
ie: 

$L = \frac{|X|}{\omega}$ if $X > 0$

OR



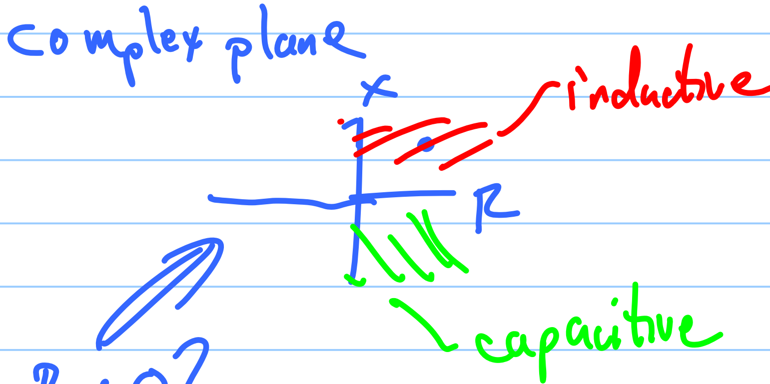
$C = \frac{1}{\omega |X|}$ if $X < 0$



(with $V = ZI$, $|V| = |Z| \cdot |I|$ and $\theta_V = \theta_Z + \theta_I$)

V is leading I
I is lagging V

inductive

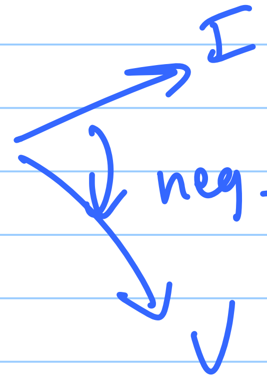


Can $R < 0$?

will see that corresponds to active ckts → power generating

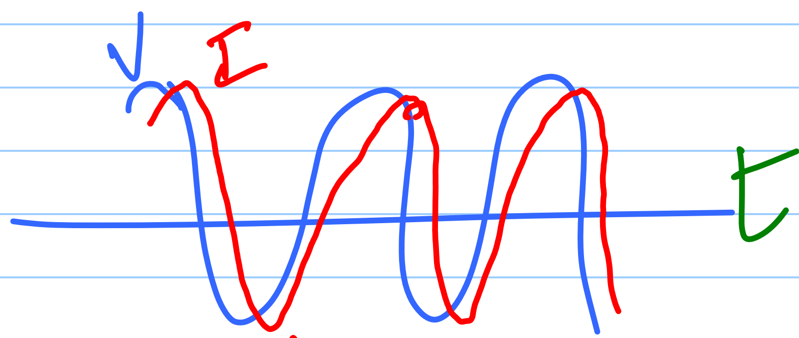
($R \geq 0$ for passive)

By convention, it is called "lagging"



neg. $\measuredangle$ if $X<0$, "leading"

- means neg. reactance
- capacitive
- V lags I, I leads V



V "gets there first", I is delayed $V \Rightarrow I$ lags V

$\Rightarrow$ inductive

We will revisit phasors later when we discuss power

FOURIER ANALYSIS — A Signal Space Perspective

preliminary remarks:

- we want to consider signals that are not comprised of sinewaves at a single frequency, but instead a mix of different frequencies
- Fourier analysis ~ represent signals as a superposition of sinewaves
- fundamental technique – express sig. in terms of "elementary" waveforms — we'll see why sinewaves are so important later
- What is signal space? The concept is to treat signals as <u>VECTORS</u>

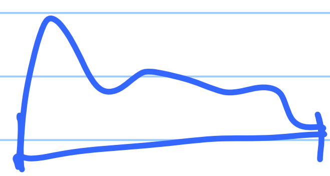
what is a vector space? $\mathcal{V}$ is a collection of objects s.t.

1. can add elements in $\mathcal{V}$
2. can multiply by scalars

} signals are vectors!!!

We start with the "classical" Fourier series

Continuous-time signals over a FINITE time interval



0 T

Consider $s(t)$ for $0 \le t \le T$ ONLY

Represent as sum of sinewaves

$\omega_0 = \frac{2\pi}{T}$ $f_0 = \frac{1}{T}$ fundamental frequency

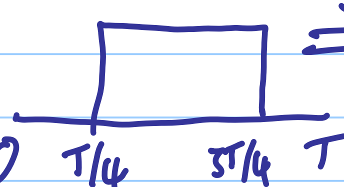
Take $\{1\} \cup \{\cos m\omega_0 t\}_{m=1}^{\infty} \cup \{\sin m\omega_0 t\}_{m=1}^{\infty}$

[means: 

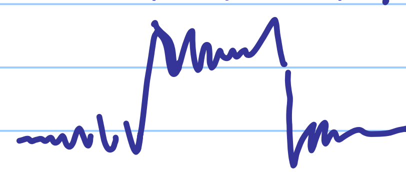
& $\cos\omega_0 t$ & $\cos 2\omega_0 t$ & .... & $\sin\omega_0 t$ & $\sin 2\omega_0 t$ & ....]

<u>Thm:</u> "all" signals can be expressed as lin. comb. of these waveforms

<u>ie:</u> $$s(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos m\omega_0 t + \sum_{m=1}^{\infty} b_m \sin m\omega_0 t$$

<u>ex:</u> If we have 

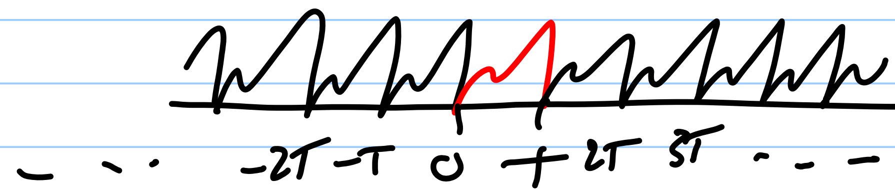
$\Longrightarrow$ Take finite # of terms in Fourier series


$\leftarrow$ Gibbs phenomenon

Note that all these sinewaves have common period T

$\therefore$ if we extend the representation beyond $0 \le t \le T$,

it is a periodic extension

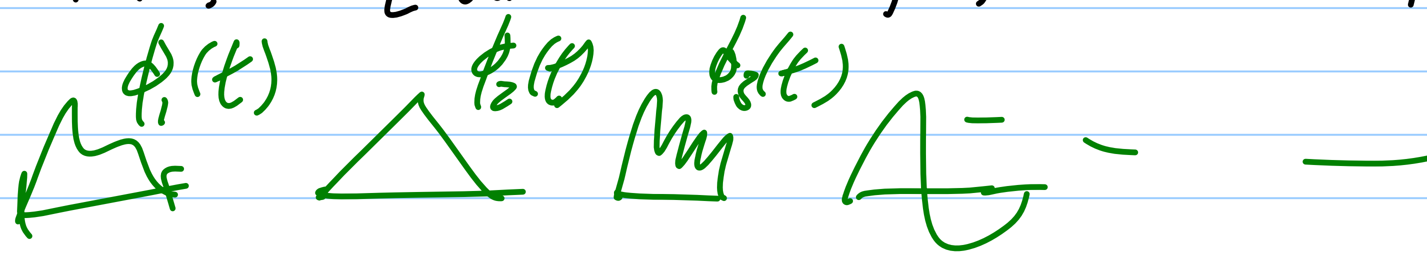


How to find the coeff? Consider a more general problem.

Suppose we have a discrete set (finite # or countably ∞) of signals $\{\phi_m(t)\}_m$, $\quad s(t) = \sum_m c_m \phi_m(t) \quad 0 \le t \le T$

Given such $s(t)$, find c_m. We may not be able to represent "all" $s(t)$.

When can we? [Would need ∞ # signals, even then may not work]



Defn: (for cont-time signals on $0 \le t \le T$)

inner product: $\langle f, g \rangle = \frac{1}{T} \int_0^T f(t) g^*(t)\, dt$ allow complex values

Properties: $\langle f, g \rangle = \langle g, f \rangle^*$

$\langle \alpha f, g \rangle = \alpha \langle f, g \rangle$ → Note: $\langle f, \alpha g \rangle = \alpha^* \langle f, g \rangle$

$\langle f_1 + f_2, g \rangle = \langle f_1, g \rangle + \langle f_2, g \rangle$

$\langle f, f \rangle = \frac{1}{T} \int_0^T |f(t)|^2\, dt$

→ so $\langle f, f \rangle \ge 0$ with $= 0$ iff $f(t) = 0$ signal (?)

other examples of inner products:

$$\langle f,g\rangle = \int_{-\infty}^{\infty} f(t)\, g^*(t)\, dt$$

$$\langle f,g\rangle = \int_{\mathbb{R}^n} f(\vec{r})\, g^*(\vec{r})\, dx_1\, dx_2 \cdots dx_n \qquad (\vec{r} = (x_1, x_2 \cdots x_n))$$

$$\langle f,g\rangle = \sum_{-\infty}^{\infty} f_n g_n^*$$

Consider $\vec{u}, \vec{v} \in \mathbb{C}^n$: $\quad \vec{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \quad \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

[by default, represent vectors as column vectors] $\quad \langle u,v\rangle = \sum_{i=1}^{n} u_i v_i^*$

$\mathbb{R}^n$: $\langle u,v\rangle = \sum_{i=1}^{n} u_i v_i = \vec{u} \cdot \vec{v}$ DOT PRODUCT!

<u>NOTE</u>: The term DOT PRODUCT reserved for $\mathbb{R}^n$. Else called inner product.

$$\langle X, Y\rangle = E(XY^*)$$

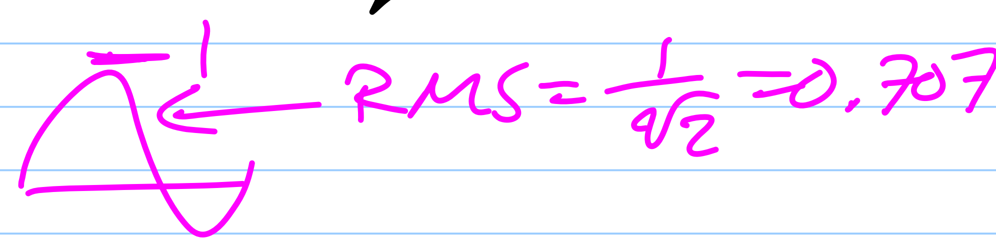
In $\mathbb{R}^n$, the length of a vector $\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}} = \sqrt{\sum u_i^2}$ $|\vec{u}|$

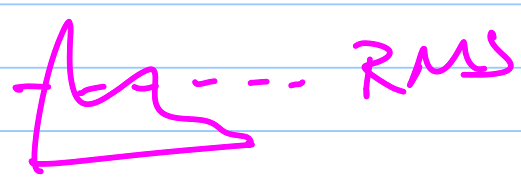
use this word for $\mathbb{R}^n$ only.

More general: norm: $\|f\| = \sqrt{\langle f, f \rangle}$
(L^2-norm)

eg: in our case: $\|f\| = \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\, dt}$

RMS value: root-mean-square





$p(t) = v(t)\, i(t)$ Watts

avg power $\rightarrow$ $P_{AV} = \frac{1}{T}\int_0^T p(t)\,dt$ inst. power

Back to phasors:

If $v(t) = |V| \cos(2\pi f_0 t + \theta)$ $\qquad \omega_0 = 2\pi/T$

$$\boxed{\begin{aligned} P_{AV} &= \tfrac{1}{2} Re(VI^*) \\ &= Re(V_{rms} I^*_{rms}) \end{aligned}}$$

Can show $\| \cos 2\pi f_0 t \| = \| \sin 2\pi f_0 t \| = \frac{1}{\sqrt{2}}$

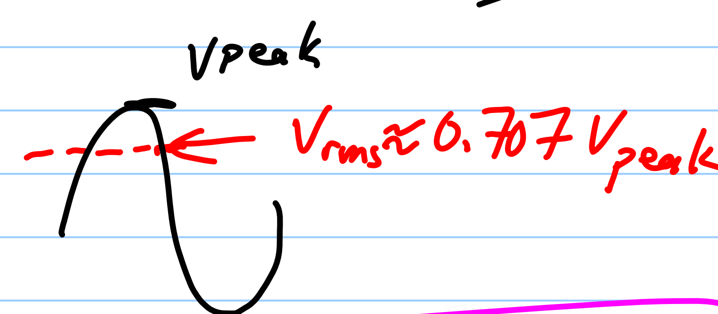
Thus, $\|v\| = \frac{1}{\sqrt{2}} |V| =$ RMS value

Define the RMS phasor as $V_{rms} = \frac{1}{\sqrt{2}} |V| e^{j\theta}$

Then $v(t) = \sqrt{2}\, Re(V_{rms} e^{j\omega t}) = \sqrt{2} |V|_{rms} \cos(2\pi f_0 t + \theta)$

and $\|v\|_{rms} = |V_{rms}|$

$|V| =$ peak value



Important: For sinewaves, $RMS = \frac{1}{\sqrt{2}} \times$ Peak

NOT $\frac{1}{\sqrt{2}}$ for other waveforms!

$$\boxed{\tfrac{1}{2} Re(\vec{E} \times \vec{H}^*)}$$

Back to inner products

L^2-norm

Properties of norm: $\|f\| \geq 0$

$\|\alpha f\| = |\alpha| \, \|f\|$ for scalar α

$\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ (triangle inequality)

$|\langle f, g \rangle| \leq \|f\| \, \|g\|$ (Schwarz inequality)

with $=$ iff f, g are scalar multiples of each other

$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$

Defn: If $\langle f, g\rangle = 0$, we say f, g are **orthogonal** and write $f \perp g$ ("f perp g")

Remark: perpendicular ($\vec{u}\cdot\vec{v}=0$) reserved for the case $\mathbb{R}^n$

Now, if $s(t) = \sum c_m \phi_m(t)$ what is $\|s\|$?

$$\|s\|^2 = \langle s, s\rangle = \left\langle \sum c_m \phi_m(t), \sum c_n \phi_n(t)\right\rangle = \sum_m \sum_n c_m c_n^* \langle \phi_m, \phi_n\rangle$$

A mess!

Simple case:
$$\|f+g\|^2 = \|f\|^2 + \|g\|^2 + \langle f, g\rangle + \langle g, f\rangle$$
$$= \|f\|^2 + \|g\|^2 + 2\,\mathrm{Re}\langle f, g\rangle$$

generalized Pythagoras!

Thus, if $f \perp g$: $\|f+g\|^2 = \|f\|^2 + \|g\|^2$

If $f_1, f_2 \cdots f_n$ mutually $\perp$: $\|f_1 + f_2 + \cdots + f_n\|^2 = \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2$

<u>Defn:</u> $\{\phi_m\}$ is an <u>orthogonal set of signals</u> if $\phi_m \perp \phi_n$ for $m \neq n$

In this case, if $s(t) = \sum c_m \phi_m(t)$ then $\|s(t)\|^2 = \sum_m |c_m|^2 \|\phi_m(t)\|^2$

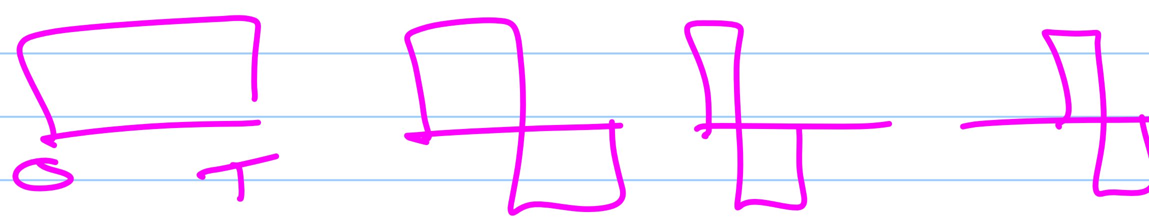
Back to the question: how to find c_m?

<u>Thm:</u> If $\{\phi_m\}$ orthogonal and $s = \sum c_m \phi_m$ then

$$c_m = \frac{\langle s, \phi_m \rangle}{\langle \phi_m, \phi_m \rangle} = \frac{\langle s, \phi_m \rangle}{\|\phi_m\|^2}$$

<u>Proof:</u> Take $s = \sum c_m \phi_m$. Then $\langle s, \phi_n \rangle = \sum c_m \langle \phi_m, \phi_n \rangle = c_n \|\phi_n\|^2 + 0$

QED



We usually work with normalized signal set:

<u>Defn:</u> $\{\phi_m\}$ is an <u>orthonormal sequence</u> (o.n.s.) if
$\phi_m \perp \phi_n$ for $m \neq n$ and $\|\phi_n\| = 1$ $\forall n$

<u>ie:</u> $\langle \phi_m, \phi_n \rangle = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$

<u>Thm:</u> $\{\phi_m\}$ o.n.s. If $s(t) = \sum_m c_m \phi_m(t)$ then

$c_m = \langle s, \phi_m \rangle$ and $\|s\|^2 = \sum |c_m|^2$ (Parseval's Thm)

This is called a generalized Fourier series
and the c_m's are called the generalized Fourier coefficients.

Thm: $\{1\} \cup \{\cos m\omega_0 t\}_{m=1}^{\infty} \cup \{\sin m\omega_0 t\}_{m=1}^{\infty}$ $(\omega_0 = 2\pi/T)$

are orthogonal over $0 \le t \le T$ with

$\|1\| = 1 \quad \|\cos m\omega_0 t\| = \|\sin m\omega_0 t\| = \frac{1}{\sqrt{2}}$, $m = 1, 2, 3, \dots$

Proof: Need to show:

$\frac{1}{T}\int_0^T 1 \cdot \cos m\omega_0 t\,dt = 0$

$\frac{1}{T}\int_0^T 1 \cdot \sin m\omega_0 t\,dt = 0$

$\frac{1}{T}\int_0^T \cos m\omega_0 t \cdot \sin n\omega_0 t\,dt = 0$

$\forall m, n \ge 1$

$\frac{1}{T}\int_0^T \sin m\omega_0 t \cdot \sin n\omega_0 t\,dt = 0 \quad (m \ne n)$

$\frac{1}{T}\int_0^T \cos m\omega_0 t \cos n\omega_0 t\,dt = 0 \quad (m \ne n)$

$\frac{1}{T}\int_0^T 1^2\,dt = 1 \quad \frac{1}{T}\int_0^T \cos^2 m\omega_0 t\,dt = 1/2$

$\frac{1}{T}\int_0^T \sin^2 m\omega_0 t\,dt = 1/2$

Tedious (but true)!

Thm: Trigonometric Fourier Series

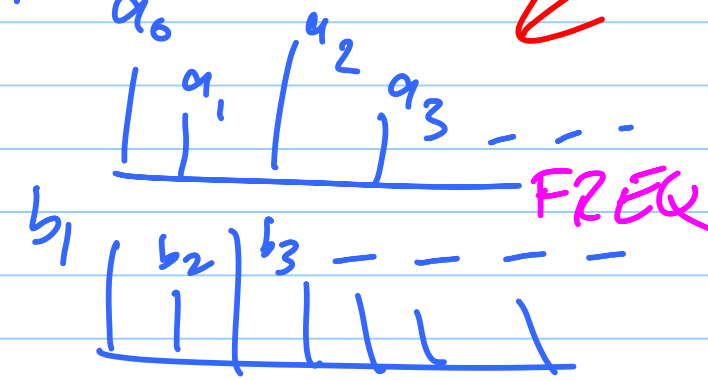
$$s(t) = a_0 + \sum_{m=1}^{\infty} a_m \cos m\omega_0 t + \sum_{m=1}^{\infty} b_m \sin m\omega_0 t$$

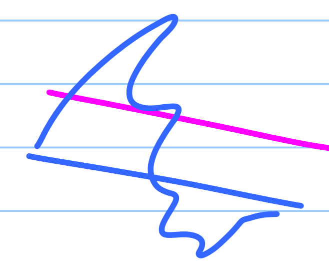
$$a_0 = \frac{1}{T}\int_0^T s(t)\,dt \qquad a_m = \frac{2}{T}\int_0^T s(t)\cos m\omega_0 t\,dt \qquad b_m = \frac{2}{T}\int_0^T s(t)\sin m\omega_0 t\,dt$$

$$\frac{1}{\|\phi\|^2} = \frac{1}{1/2} = 2$$

Spectrum

Graph of a's & b's ← line spectrum





a_0 = DC component

$$\frac{1}{T}\int_0^T s(t)\,dt$$

$\left.\begin{matrix} a_1, a_2 \cdots \\ b_1, b_2 \cdots \end{matrix}\right\}$ AC components

$f_0 = \frac{1}{T}$ fundamental frequency

$m \cdot f_0 = m^{th}$ harmonic

Complex exponential form

$m \in \mathbb{Z}$ (pos & neg val.)

$\phi_m(t) = e^{jm\omega_0 t}$, $\omega_0 = 2\pi/T$

$\langle \phi_m, \phi_n \rangle = \frac{1}{T}\int_0^T e^{jm\omega_0 t} e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_0^T e^{j(m-n)\omega_0 t}\, dt$

$\underline{m=n}$: $\frac{1}{T}\int_0^T 1\, dt = 1$ $\quad \|\phi_m\| = 1$

$m \neq n$: $\frac{1}{T} \frac{1}{j(m-n)\omega_0} e^{j(m-n)\omega_0 t}\Big|_0^T = 0$

$\{\phi_m\}_{m=-\infty}^{\infty}$ is an o.n.s.

$e^{j2\pi[\text{INTEGER}]} = 1$

$$\boxed{\begin{aligned} s(t) &= \sum_{-\infty}^{\infty} c_m e^{jm\omega_0 t} \\ c_m &= \frac{1}{T}\int_0^T s(t)e^{-jm\omega_0 t}\,dt \end{aligned}}$$

$$\|s\|^2 = \sum_{-\infty}^{\infty} |c_m|^2 = \frac{1}{T}\int_0^T |s(t)|^2\,dt$$

"Power" average

$$\|s\|^2 = a_0^2 + \frac{1}{2}\sum_{m=1}^{\infty}\underbrace{(a_m^2 + b_m^2)}_{\text{power in } m^{th} \text{ harmonic}}$$

Power in m^{th} harmonic

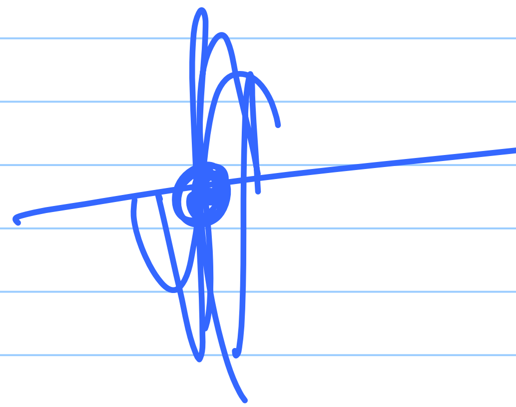
$$\frac{1}{2}(a_m^2 + b_m^2) = |c_m|^2 + |c_{-m}|^2$$

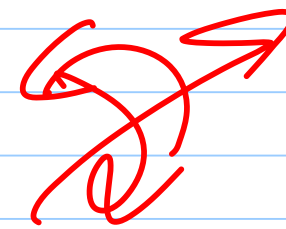
DC power: $a_0^2 = |c_0|^2$

$$c_m = \frac{a_m - j b_m}{2} \qquad c_{-m} = \frac{a_m + j b_m}{2}$$

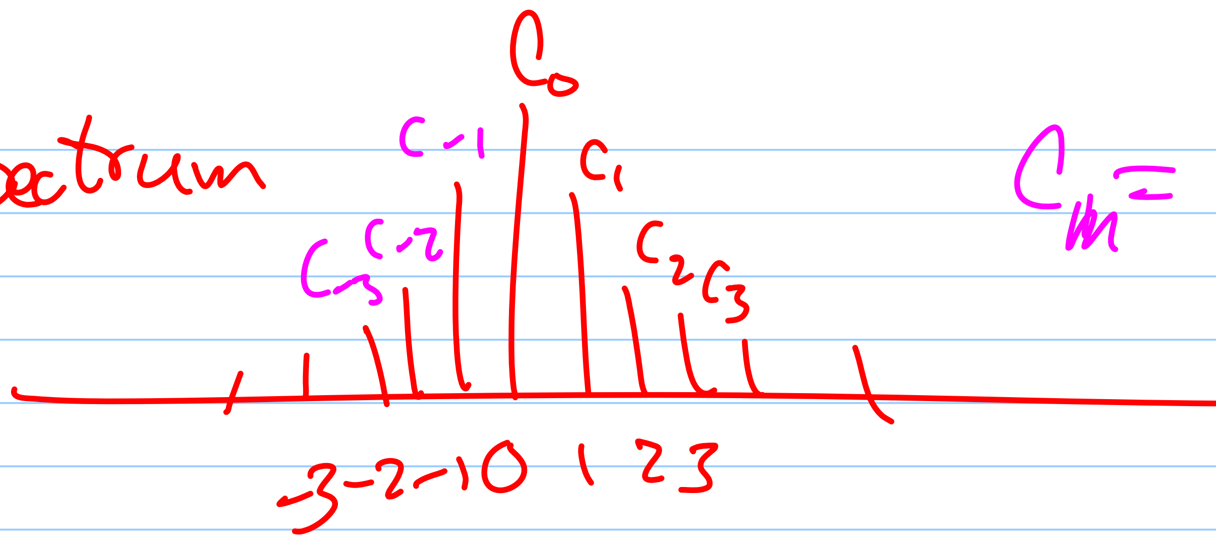
(real signals) $\quad c_m = c_{-m}^*$

$$a_m = 2\,\mathrm{Re}(c_m) \qquad b_m = -2\,\mathrm{Im}(c_m)$$





line spectrum



$c_m = c_{-m}^{*}$

$\{\phi_m\} \Longrightarrow \overset{\text{Given}}{s} = \sum c_m \phi_m$: find c's

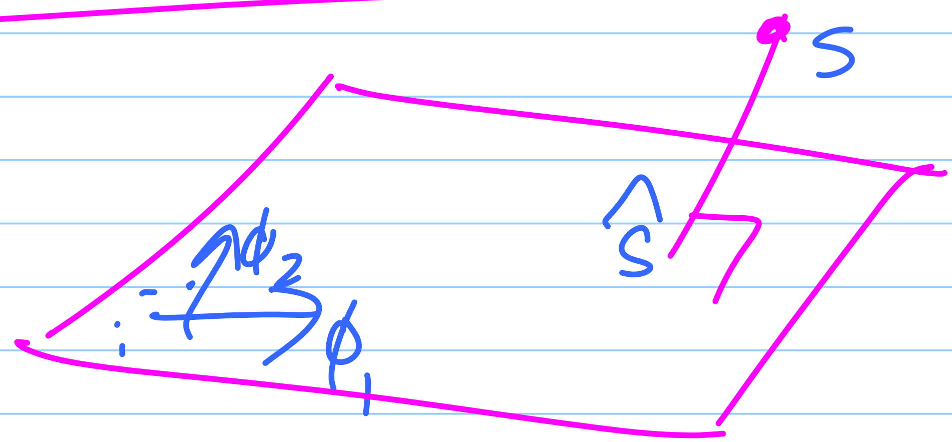
Given $s(t)$, can we write $s = \sum c_m \phi_m$?

Complete o.n.s: every $s(t)$ can be represented

$\{\phi_m\}$ o.n.s, $\quad s(t)$

Find c_m that best approximate $s(t)$

$\hat{s}(t) = \sum c_m \phi_m(t)$: $\|s - \hat{s}\|$ is min.



THM

$$C_m^{OPT} = \langle s, \phi_m \rangle$$

Corollary: $\hat{s} = \sum c_m \phi_m(t)$ $\min \|s - \hat{s}\|$

iff $s - \hat{s} \perp \phi_m \quad \forall m$

ORTHOGONALITY PRINCIPLE

$$\langle s - \hat{s}, \phi_m \rangle = \langle s, \phi_m \rangle - \langle \hat{s}, \phi_m \rangle$$

$$= \langle s, \phi_m \rangle - C_m^{OPT}$$

$$= 0$$

$$\hat{s}(t) = \sum c_m^{opt} \phi_m(t)$$

$$\sum \alpha_m \phi_m(t) = \tilde{s}(t) \qquad c_m^{opt} = \langle s, \phi_m \rangle$$

$$\tilde{s}(t) = \sum (\alpha_m - c_m^{opt}) \phi_m(t) + \sum c_m^{opt} \phi_m(t)$$

$$s(t) - \tilde{s}(t) = (s(t) - \hat{s}(t)) + \underbrace{(\hat{s}(t) - \tilde{s}(t))}_{\text{lin comb of } \phi\text{'s}}$$

$$c_m^{opt} = \langle s, \phi_m \rangle$$

$$\perp \phi_m$$

$$\|s - \tilde{s}\|^2 = \|s - \hat{s}\|^2 + \cancel{\sum |\alpha_m - c_m^{opt}|^2} \quad 0$$

$$s(t) = \big(s(t) - \hat{s}(t)\big) + \sum c_m \phi_m(t)$$

$\perp$ all ϕ's

$\langle s, \phi_m \rangle$

$$\|s\|^2 = \|s - \hat{s}\|^2 + \sum |c_m|^2$$

Bessel's Inequality: $\boxed{\|s\|^2 \geq \sum_m |\langle s, \phi_m \rangle|^2}$

complete ons: $\|s\|^2 = \sum |\langle s, \phi_m \rangle|^2$

Parseval's Identity