# A Distributed Numerical Approach for Managing Uncertainty in Large-Scale Multi-agent Systems

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**Abstract.** Mathematical models of complex processes provide precise definitions of the processes and facilitate the prediction of process behavior for varying contexts. In this paper, we present a numerical method for modeling the propagation of uncertainty in a multi-agent system (MAS) and a qualitative justification for this model. We discuss how this model could help determine the effect of various types of uncertainty on different parts of the multi-agent system; facilitate the development of distributed policies for containing the uncertainty propagation to local nodes; and estimate the resource usage for such policies.

## 1 Introduction

Agents operating in open environments should be capable of handling the uncertainty in there environments in order to provide performance guarantees. This includes determining and controlling the source and propagation of the uncertainty. Mathematical models of complex processes provide precise definitions of the processes and facilitate the prediction of their behavior for varying contexts. In this paper, we present a closed-form numerical method for modeling the propagation of uncertainty in a multi-agent system (MAS) and a qualitative justification for this model. We discuss how this model could help determine the effect of various types of uncertainty on different parts of the multi-agent system; facilitate the development of distributed policies for containing the uncertainty propagation to local nodes; and estimate the resource usage for such policies.

We will first describe the characteristics of the MAS problem we are interested in addressing and then model this problem using a numerical Partial Differential Equation. The solution to this particular Partial Differential Equation (PDE) will describe the behavior of the MAS for varying number of agents and for varying amounts of time. In keeping with the philosophy behind derivations of PDEs governing physical processes, we derive the PDE using only small time intervals

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and only for agents with direct dependencies to other agents. The advantage of this procedure is that as soon as such a model is derived, the solution of that PDE provides the prediction of system behavior for all times (small or large) and for different kinds of agent networks. Providing information to predict performance degradation in individual agents due to the propagation of the uncertainty will facilitate the construction and implementation of contingent plans to control the propagation of uncertainty and contain its potential negative effects on the performance characteristics of the cooperative MAS.

In order to apply the mathematical techniques to the study of propagation of uncertainty, we consider large scale multi-agent systems (MAS) that are nearly-decomposable and have hundreds of heterogeneous agents. Simon [10] defines a nearly-decomposable system as a system made up of sub-systems where the interactions between sub-systems are weak and they behave nearly independently. This implies that inter-subsystem interactions are weaker than intra-subsystem interactions. In our work, we assume that most of these interactions can be predicted at design time [5]. Another simplification is that we abstract the details of uncertainty propagation into a probability per unit time for an individual to be influenced by uncertainty. This abstraction will allow for the construction of the PDE-based model.

Figure 1 describes a nearly-decomposable system that we will use to describe the issues addressed by this paper. The nearly-decomposable multi-agent system has a large number of cooperative agents. We assume that each agent is capable of performing a set of tasks. *Agent interactions* are defined as those relationships where the performance characteristics (utility, duration, cost) of a task(s) in

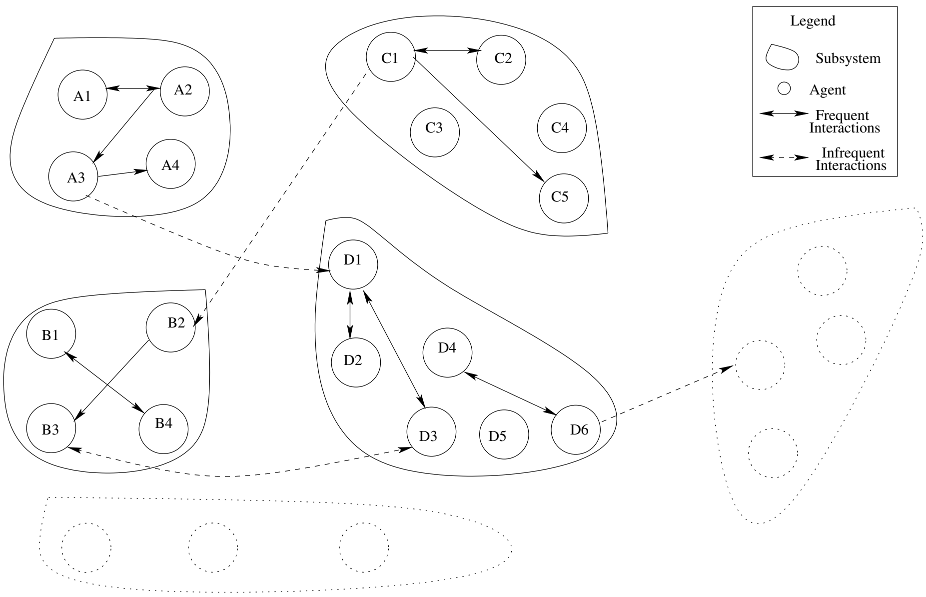


**Fig. 1.** Nearly-Decomposable Multi-agent System

one agent affect the performance characteristics of task in another agent. In other words, uncertainty in the execution of a task in one agent might have adverse effects on the execution of tasks in other agents. An example of such an interaction would be an *enables* relationship [4] where successfully completing a task (or a primitive action) in one agent is a pre-condition to initiating a related task (primitive action) in another agent. In Figure 1, there are a number of agents that have frequent interactions, for instance agent sets $\{A1, A2, A3, A4\}$, $\{B1, B2, B3, B4\}$, etc. Agents with frequent interactions are grouped loosely together into sub-systems. There are other sets of agents $\{A3, D1\}$, $\{B2, C1\}$, $\{B3, D3\}$ that exhibit infrequent interactions.

There is significant previous research in reasoning about uncertainty in multi-agent systems [1,3,9,12,13]. Many of these use formal models and decision theoretic approaches to describe and reason about uncertainty in scheduling and coordination. The bottleneck in most of these approaches is that as the number of agents and interactions increase, determining the optimal solution becomes very difficult. Hence many of these approaches use approximate techniques to compute the solution. Our work is novel in that we are use a linear mathematical model to reason about uncertainty and the advantage of a closed-form approach is that it scales to large MAS. This mathematical model has also used to reason about other complex processes like heat conduction in materials [2] and propagation of viruses in computer networks [6]. Uncertainty propagation in MAS under certain assumptions is analogous to propagation of computer viruses in computer networks. In this paper, we limit ourselves to studying the uncertainty of an agent completing a task successfully where that task is a pre-condition to the task belonging to another agent. These are also called *enables* relationships [4] Computer viruses propagate through networks causing node failures while uncertainty/failure propagates through MAS networks causing agents to fail in their tasks. We are currently working on extending this model to the propagation of different levels of uncertainty from one agent to another. An example of such an interaction would be the *facilitates* relationship [4] where if some task A facilitates a task B, then completing A before beginning work on B might cause B to take less time, or cause B to produce a result of higher quality, or both. If A is not completed before work on B is started, then B can still be completed, but might take longer or produce a lower quality result than in the previous case. In other words, completing task A is not necessary for completing task B, but it is helpful.

The paper is structured as follows: In Section 2, we present the issues involved in modeling the problem using a PDE, the derivation of a PDE model of uncertainty propagation in nearly-decomposable MAS and a qualitative analysis of the model. In Section 3, we summarize the results and discuss future work.

## 2 The Model

In this section, we describe the process of modeling the uncertainty propagation problem as a Partial Differential Equation (PDE) as well as the benefits of using a PDE. The main difference between PDE and Ordinary Differential

Equation(ODE)-based mathematical models is that the first one is distributed, whereas the second one is non-distributed. An ODE-based model is used when a certain instance is treated as a "non-divisible whole". For example, if one wants to describe the acceleration of a car by modeling the whole car as a moving particle, one uses the ODE $\frac{d^2x}{dt^2} = a(t)$, where $t$ is time, $x(t)$ is the travel distance and $a(t)$ is the acceleration. However, if one wants to describe processes taking place inside of certain parts of the car during its run, e.g., vibration, then one should use PDEs. Analogously, in the case of multi-agent systems, ODE-based models describe all influenced agents as a *non-divisible* subset of the entire network. The distributed PDE-based models, on the other hand, differentiate between different influenced agents.

A PDE is described as a continuous function in euclidean space. So how do we get a continuous model from a discrete set of agents and their interactions? There is a well-known procedure [2] used in the derivations of PDEs that govern physical phenomena in nature where a continuous model is obtained from a discrete model, provided that the number of discrete points is large. This assumption holds here since we consider only large-scale MAS in this work. The procedure involves uniting a subset of agents (subject to some rules)[1] into a set and assigning a point on the real line to this set. Next, we consider another set of agents, which are in a particular kind of relation with the first one and assign a close point on the real line to this set. Continuing this process, we obtain a grid of points with the step size $h$ on the real line. A close analog is a grid in a finite difference scheme [7] for the numerical solution of a PDE. Next, we assume that all points in the gaps between grid points also represent similar sets of agents. This produces a continuum of points. The continuous functions in this paper can all be viewed as interpolations of corresponding discrete functions. This derivation procedure allows for a continuous model that is often easier to analyze theoretically than its discrete counterpart. It turns out that a proper theoretical analysis can often lead to optimal numerical schemes.

Although, the continuous model is more convenient for the theoretical analysis, finding solutions for continuous PDEs is computationally intractable except for trivial cases. Trivial cases are those PDEs with constant coefficients where the solutions are given by explicit formulas. So, the concept of finite differences [7] is used to solve the PDE. A main idea behind finite differences is that the derivative of a function $U(x)$ is $\frac{U(x+h)-U(x)}{h}$ (approximately), where h is the grid step size. Thus, the derivatives involved in a PDE can be approximated by discrete finite differences. This produces an algebraic system of equations. The solution to this algebraic system is the approximation of the exact solution of the corresponding PDE at those grid points.

In principle, if the grid step size h approaches zero, the solution of that algebraic system approaches the solution of the continuous model. This is because the derivatives are approximated more accurately as h approaches zero. But when h decreases, it increases the size of the algebraic system. This could potentially

[1] The subsystems which are part of the nearly-decomposable MAS could provide a natural mapping to these agent sets.

bring instabilities in the solution. The advantage of the continuous PDE is that it guarantees that such instabilities will not occur, specifically it does not occur for the PDE's discussed in this paper. In summary, we use a continuous model because the theory is well-defined and it helps in the derivation of suitable finite difference schemes, which are stable and the solutions of corresponding algebraic systems accurately approximate solutions of those continuous PDEs . Our goal is to perform the following sequence of steps:

1. Embed the original discrete math model into a continuous diffusion PDE;
2. Solve this PDE by defining a finite difference grid with grid step size h.

The model is based on the following three conjectures which are analogs of the Fickian diffusion law, the Fourier law of heat conduction and the conservation law. In classical settings, conjectures about natural phenomena have been always validated experimentally, which led to verifications of corresponding mathematical models. This paper only presents the plausibility of the conjectures and we plan to validate the conjectures experimentally as future work.

### 2.1 Uncertainty Propagation in MAS Using a Continuous Model

As described earlier, our model works under the assumption that for a given agent $Y$ in the MAS, there are a certain set of agents $Z$ that are influenced by the uncertainty in agent $Y$'s actions.

Let $N$ be the total number of agents in the MAS. To construct the continuous model, we take a subset of agents $M_1 \subset N$ of this set. The set $M_1$ is represented by a point $x_1$ on the line of real numbers. Next, we take the set $M_2$ of all agents whose uncertainty is influenced by $M_1$. The set $M_2$ is represented by a point $x_2 > x_1$. Next, take the set $M_3$ of all agents, that are influenced by the agents in the set $M_2$. $M_{1,2}$ is the set of all agents in $M_1$ that might be influenced by $M_2$, i.e., influence can go both ways: from $M_1$ to $M_2$ and backwards. We do not include $M_{1,2}$ in $M_3$ in order to guarantee that each agent is in exactly one set. The set $M_3$ is represented by a point $x_3 > x_2$. Next, take the set $M_4$ of all agents, which are influenced by agents in the set $M_3$, but not influenced by those agents in the set $M_{1,3} \cup M_{2,3}$. The set $M_4$ is represented by a point $x_4 > x_3$.

This process is repeated assuming that the distance between neighboring points is the same, i.e., $x_i - x_{i-1} = h$, where $h$ is a small positive number independent of the point $x_i$. So, there are $n >> 1$ points $\{x_i\}_{i=1}^{n}, -\infty < x_1 < x_2 < x_3 < ... < x_n < \infty$ and

$$\bigcup_{i=1}^{n} M_i = N \tag{1}$$

These set of points describe an embedded continuous model. The interval $x \in [a, b]$ contains representatives of sets of agents. This is analogous to the Finite Difference Method for numerical solutions of PDEs/ODEs. While the embedded continuous model, is certainly convenient for the theory, in the case of numerical solutions we can consider points $x_1, x_2, ..., x_n$ with $x_i - x_{i-1} = h$ as grid points of a corresponding finite difference scheme. Thus, we will actually find the solution of the corresponding PDE at these points. It is also well-known that a solution for a system of finite difference equations is often interpolated on the entire interval as a sufficiently smooth function.

Suppose $N(x_i)$ is the number of agents in the set $M_i$ and $N_{inf}(x_i, t)$ is the number of agents influenced by uncertainty in this set at the moment of time $t$. Then

$$u(x_i, t) = \frac{N_{inf}(x_i, t)}{N(x_i)} << 1 \tag{2}$$

is the relative number of influenced agents in the set $M_i$. Near-decomposability of the MAS implies that the uncertainty from some agents of $M_i$ might infect only neighboring agents: either those from the set $M_{i+1}$ or those from the set $M_{i-1}$. In other words, an instantaneous change in the relative number $u(x_i, t)$ of influenced agents of the set $M_i$ at the time $t$ will at most cause an instantaneous change of $u(x_{i-1}, t)$ and $u(x_{i+1}, t)$.

Therefore, in the continuous model, we assume that for a *sufficiently small* time interval, agents of the set $M_x$ can be influenced only by agents in very close sets $M_y$ with $y \approx x$. This means that a change of the function $u(x, t)$ at a point $x$ and at a moment of time $t$ causes a change of this function only for $y \approx x$ for times $t' \approx t$. However, this does not preclude a change of the function $u(x, t)$ at a point $x$ to cause changes of $u(y, t'')$ for points $y$ located far from the point $x$, as long as the moment of time $t''$ is far from $t$ and $t'' > t$.

All functions involved in the derivations below are assumed to be sufficiently smooth. For each point $y \in [a, b]$ and for each time moment $t > 0$, let $M_y$ be the set of all agents represented by the point $y$, $N(y)$ be the total number of agents in $M_y$. Also, let $\Delta x$ and $\Delta t$ be two sufficiently small positive numbers, i.e., $0 < \Delta x, \Delta t << 1$. Then, one can derive [6], the *density of influence of the interval* $[y, y + \Delta x]$ at the moment of time $t$ to be

$$\frac{N(y + \Delta x)u\,(y + \Delta x, t) - N(y)u\,(y, t)}{\Delta x} \tag{3}$$

The density of influence is approximately the average change, over the interval $[y, y + \Delta x]$, of the number of agents influenced by the uncertainty at the moment of time $t$. In a small time interval $(t - \Delta t, t + \Delta t)$, agents belonging to sets $M_x$ with points $x$ located in the interval $[y, y + \Delta x]$ might be influenced only by agents from sets $M_z$ for $z \approx y \approx y + \Delta x$. And vice versa: in a small time interval $(t - \Delta t, t + \Delta t)$ agents belonging to these sets $M_z$ might influence only agents of sets $M_x$ with $x \approx y \approx y + \Delta x$.

Taking the limit for the above equation, the *density of influence* at the point $x$ and at the moment of time $t$ can be derived as

$$N(x)\frac{\partial u}{\partial x}\,(x, t) \tag{4}$$

Let $G\,(x, t, t + \Delta t)$ be the number of hosts, which include: (1) agents influenced by agents of the set $M_x$ in a small time interval $(t, t + \Delta t)$ plus (2) those agents in the set $M_x$, which are influenced by agents from neighboring sets $M_z$ with $z \approx x$ during the same time. $G(x, t, t + \Delta t)$ is proportional to the density of influence,

$$G(x, t, t + \Delta t) = D(x)N(x)\frac{\partial u}{\partial x}(x, t)\, \Delta t \tag{5}$$

where the diffusion coefficient $D(x)$ is an analog of the influence rate at the point $x$, i.e., $D(x)$ is the relative number of agents in an infinitesimally small period of time, that are influenced by uncertainty in agents of the set $M_x$ plus those of the set $M_x$, which are influenced by agents in neighboring sets. $D(x)$ is hence related to the *connectivity* averaged over all agents in the set $M_x$. It will be shown below that $D(x)$ is an analog of the *diffusion coefficient* and that the above equation is an analog of the Fickian diffusion law [8].

In summary, $G(x, t, t + \Delta t)$ is the *influence* through the point $x$ in the time interval $(t, t + \Delta t)$ where $D(x) \geq constant > 0$.

We now describe an analog to our model based on the 3-dimensional version of the Fourier law of heat conduction [11]. Let $V \subset R^3$ be a certain volume, $S$ be its boundary and the function $v(X, t)$ be the temperature at the point $(x, y, z) = X \in R^3$ at the moment of time $t$. Then the heat flow across the closed surface $S$ in the small time interval $(t, t + \Delta t)$ is

$$Q = \Delta t \int_S k\frac{\partial v}{\partial n} dS, \tag{6}$$

where $n(X)$ is the unit outward normal vector on $S$ and $k(X)$ is the heat conduction coefficient, which is an analog of the diffusion coefficient. Now consider the 1-dimensional version of this formula. Specifically, in the 1-dimensional case, the interval $[y, y + \Delta x]$ is taken instead of the volume $V$ and $S$ is replaced with two edge points $y$ and $y + \Delta x$. We have then $n(y + \Delta x) = 1$ and $n(y) = -1$.

The conservation law, which states that "*the rate of change of the amount of material in a region is equal to the rate of change of flow across the boundary plus that which is created within the boundary*" [8]. By analogy with the conservation law, it is shown [6] that the following equation can be derived

$$\int_y^{y+\Delta x} \frac{\partial}{\partial x}\left(D(x)N(x)u_x(x, t)\right) dx = \int_y^{y+\Delta x} N(x)\frac{\partial u}{\partial t}(x, t) dx. \tag{7}$$

Suppose $F(x, t)$ is the *level of impact* of individual uncertainty at the point $x$ at the time $t$. If an agent's uncertainty is completely due to the propagation of uncertainty from another agent, then the impact is due to an external uncertainty and $F(x, t)$ is set zero. If the uncertainty in an agent is due to its own actions, then $F(x, t) > 0$.

The following derivation of the **diffusion equation** is obtained as shown in [6] :

$$N(x)u_t = \left(D(x)N(x)u_x\right)_x + N(x)F(x, t). \tag{8}$$

where $N(x)$ is number of agents in $M_x$; $u_t$ is the derivative of the number of influenced agents with respect to time t; $u_x$ is the derivative of the number of influenced agents with respect to point x; $D(x)$ is the diffusion coefficient at the point x which is related to the *connectivity* averaged over all agents in the set $M_x$ ; $F(x,t)$ is the level of impact of individual uncertainty. This equation captures the conservation law in terms of uncertainty where the rate of change of the amount of uncertainty in a subsystem is equal to the rate of change of of uncertainty flowing into the subsystem plus that which is created within the subsystem.

The initial condition for the diffusion equation is

$$u(x,0) = A\delta\left(x - x_0\right), \tag{9}$$

where $A$ is a positive number. This initial condition reflects the fact that the source of the uncertainty was in the set $M_{x_0}$ and it had the power $A$.

If $N(x)$ is a constant, then the diffusion equation is simplified as

$$u_t = \left(D(x)u_x\right)_x + F(x,t). \tag{10}$$

### 2.2 Brief Analysis of the Model

We plan to perform a careful experimental analysis of this model as future work. We now present a brief qualitative analysis instead. Suppose we assume the diffusion coefficient is a constant, thereby considering a trivial case of the PDE i.e. for $x \in (-\infty, \infty), D(x) = constant > 0, F(x,t) = 0$. Then the influence of uncertainty at time $t$ is the solution to the diffusion equation defined above and is given by

$$u(x,t) = \frac{A}{2\sqrt{\pi D t}} \exp\left[-\frac{(x-x_0)^2}{4Dt}\right] \tag{11}$$

In the above solution, the decay of the function $u(x,t)$ is slower for larger values of $D$. This means that the larger the diffusion coefficient, i.e. the larger the connectivity among the agents, the wider the influence of the uncertainty at any fixed moment of time $t$.

Also, for any fixed value of $x$, the function $u(x,t)$ increases first for small values of $t$ and then decreases to zero starting at a value $t = t(x)$,. The initial increase is due to influence in the sets $M_y$ for values of $y$ close to $x$. Let, for example $x > x_0$. Then $x$ feels the spread of the uncertainty effects, which comes from the left, because $x$ is located to the right of the initial source of the uncertainty $x_0$. Also, $x$ begins to "sense" the influence of uncertainty at the moment of time $t = t(x)$, where $t(x)$ is such a value of $t$, at which the function $u(x,t)$ becomes significantly different from zero: also, $\lim_{t\to 0} u(x,t) = 0$ for $x \neq x_0$. This supports our intuition of how uncertainty influences other agents in a nearly-decomposable system where one sub-system completes its task before another begins: the uncertainty in the initial sub-system influences agents of a certain set. This set, in turn influences "neighboring" sets, and so on. But after a while, no new influence comes to the set $M_x$ and the relative number of influenced agents eventually becomes negligibly small.

## 3 Conclusions and Future Work

In this paper, we have derived a Partial Differential Equation-based model for uncertainty propagation in large-scale multi-agent systems. We have used the similarities between uncertainty propagation and Fourier law of heat conduction to derive a model that is scalable and linear. The model helps determine the spread of influence of uncertainty over time and can help determine performance degradation caused by the uncertainty. It also gives us information on how best to control the uncertainty by using potential contingency plans which keep track of the possible path of the propagation. In this paper, we considered the case where uncertainty affects hard interactions such as *enables* between agents.

There are number of exciting areas of future work. We first plan to validate this model experimentally in a simulation system. The experiments will validate and help modify the model first for the hard interactions described in this paper, then for soft interactions such as *facilitates* and finally the aggregation of uncertainty as it propagates through the MAS. We also will use the simulations to derive reasonable bounds for coefficients of the diffusion PDEs and their derivatives for networks of varying sizes and with varying interactions.

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